Strict Local Martingales, Random Times, and Non-Standard Changes of Probability Measure in Financial Mathematics

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Dörte Kreher

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Deutschland

Promotionskomitee

Prof. Dr. Ashkan Nikeghbali (Vorsitz)
Prof. Dr. Jean Bertoin

Prof. Dr. Monique Jeanblanc (Gutachterin)
Prof. Dr. Philip Protter (Gutachter)

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Zusammenfassung

Die vorliegende Arbeit befasst sich mit nicht gewöhnlichen Wechseln von Wahrscheinlichkeitsmaßen in Verbindung mit strikt lokalen Martingalen oder beliebigen Zufallszeiten, die durch Probleme in der Finanzmathematik motiviert sind.
Zuerst untersuchen wir Finanzblasen in Aktienkursen, die durch strikt lokale Martingale modelliert werden. Um den Einfluss von Finanzblasen auf die Bewertung von Derivaten zu bestimmen, konstruieren wir zunächst ein neues Wahrscheinlichkeitsmaß mit Hilfe eines strikt positiven, strikt lokalen Martingales mit càdlàg Pfaden. Anschließend leiten wir Zerlegungsformeln her für die Preise bestimmter Klassen von pfadabhängigen europäischen Optionen und Last-Passage-Time-Formeln für europäische und amerikanische Tauschoptionen, wenn die Basiswerte der Optionen Finanzblasen aufweisen. Außerdem schlagen wir eine neue Art der Filtrationsvervollständigung entlang einer Folge von Stoppzeiten vor, die für Maßwechsel mit Hilfe von strikt lokalen Martingalen geeignet ist.
Danach untersuchen wir Maßwechsel auf Intervallen, die durch Zufallszeiten, die keine Stoppzeiten sind, begrenzt sind. Wir benutzen Methoden aus der allgemeinen Theorie der stochastischen Prozesse sowie der Theorie der progressiven Filtrationserweiterung, um eine detaillierte Analyse für sogenannte honest times und Pseudo-Stoppzeiten vorzunehmen. Zudem behandeln wir die Frage nach der Arbitragefreiheit eines Finanzmarktes auf einem zufälligen Zeithorizont. Unter der Annahme, dass der Finanzmarkt die No-Free-Lunch-with-Vanishing-Risk-Bedingung bezüglich der kleineren Filtration erfüllt, leiten wir eine hinreichende Bedingung für die Existenz eines risikoneutralen Maßes in der erweiterten Filtration her, welche auf der multiplikativen Zerlegung des Azéma-Supermartingals beruht.

Abstract

In this thesis we study non-standard changes of probability measure in relation with strict local martingales or arbitrary random times motivated by problems in financial mathematics.
First, we do an analysis of asset price bubbles modeled by strict local martingales. In order to determine the influence of asset price bubbles on the pricing of derivatives we construct a new probability measure associated with a càdlàg strictly positive strict local martingale. This allows us to derive decomposition formulas for the prices of certain classes of European path-dependent options and last passage time formulas for the prices of European and American exchange options written on underlyings with bubbles. Moreover, we introduce a new kind of augmentation of filtrations along a sequence of stopping times which is suitable for a change of measure by a strict local martingale.
Second, we study changes of probability measure up to random times which are not stopping times. Using techniques from the general theory of stochastic processes and the theory of progressive enlargement of filtrations, the cases of honest times and pseudo-stopping times are discussed in detail. We also address the question of no arbitrage up to a random time. Assuming that the market satisfies No Free Lunch with Vanishing Risk in the smaller filtration we derive a sufficient condition in terms of the multiplicative decomposition of the Azéma supermartingale for the existence of a risk-neutral measure up to a random time in the enlarged filtration.

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Contents

Chapter 1

Introduction

A widely used model for a financial market is given by a positive d-dimensional stochastic process on a filtered probability space. The stochastic process normally describes the asset price dynamics, while the filtration on the other hand represents the information structure in the market. Classical questions from mathematical finance are then, whether the model is free of arbitrage, and how to price and hedge derivatives. Concerning the first question one has to specify the precise no arbitrage condition. In general continuous time financial market models without frictions the condition of no free lunch with vanishing risk (NFLVR) is most frequently used, because it allows for a nice version of the first fundamental theorem of asset pricing, cf. Section 1.2.1. This theorem states that NFLVR is in fact equivalent to the existence of an equivalent local martingale measure also called risk-neutral measure. Under this measure the discounted stock price process turns into a local but not necessarily a true martingale. As we will see in section 1.2.3 the question of whether it is a true martingale or a strict local martingale is related to the existence of asset price bubbles. Moreover, if an equivalent local martingale measure exists, its Radon-Nikodym density process forms a uniformly integrable martingale. However, in some models the candidate deflator process turns out to be only a local martingale and therefore it does not define an equivalent local martingale measure.
While changes of probability measure find its way into the field of mathematical finance via its connection to no arbitrage conditions, changes of filtration are used to model different information levels. The theory of enlargement of filtrations has proven to be a powerful tool for the analysis of credit risk models and insider information. There are two ways of adding additional information into a given filtration. Either one assumes that the additional information is already known at time $t = 0$ (initial enlargement) or that the agent learns about it as time evolves (progressive enlargement). The second case is of course more complicated and - apart from a few other specific cases - only the case of progressive enlargement with a random time has been treated extensively in the literature. It is known that for an arbitrary random time σ the semimartingale property is in general only preserved until the time σ itself in the progressively enlarged filtration.
As in general neither strict local martingales nor market models with random time horizons do allow for a standard change of probability measure, new techniques from stochastic analysis are required if one wants to answer questions of no arbitrage and derivative pricing in these models. In this thesis I therefore investigate non-standard changes of probability measure either via strict local martingales or up to random times as well as applications thereof in mathematical finance.
In the remaining part of this first chapter a review of basic notions and results about stochastic processes and the mathematical theory of financial markets is given. The chapter ends with a short overview of Chapters 2, 3, and 4.

1.1 Some concepts in stochastic analysis

We briefly introduce some basic but important concepts from stochastic analysis repeatedly used throughout the thesis. For more information we refer to the textbooks [65] and [67].

1.1.1 Local martingales

Definition 1.1.1. A right-continuous adapted process $M = (M_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ is

i) a martingale if it is integrable and $\mathbb{E}^{\mathsf{P}}(M_t|\mathcal{F}_s) = M_s$ a.s. for all $0 \leq s \leq t < \infty$,

ii) a uniformly integrable martingale if there exists an integrable $\mathcal{F}$-measurable random variable M_∞ such that $\mathbb{E}^{\mathsf{P}}(M_\infty|\mathcal{F}_s) = M_s$ a.s. for all $s \geq 0$,

iii) a local martingale if there exists a sequence of increasing stopping times (τ_n) with $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that $\left(M_{t\wedge\tau_n}\mathbb{1}_{\{\tau_n>0\}}\right)_{t\geq 0}$ is a uniformly integrable martingale for each $n \in \mathbb{N}$.

Throughout the thesis we will only deal with local martingales M for which M_0 is integrable, in which case M^{τ_n} is a uniformly integrable martingale for each n. Moreover, every non-negative local martingale with $M_0 \in L^1$ is a supermartingale by Fatou's lemma, and it is a martingale if and only if its expectation process is constant. In Chapter 2 and Chapter 3 we will be interested in strictly positive local martingales which fail to be true martingales.

Definition 1.1.2. A local martingale which is not a true martingale is called a strict local martingale.

The original example of an integrable strict local martingale is due to [45] and is known as the reciprocal three-dimensional Bessel process.

Example 1.1.3. Let W be a three-dimensional Brownian motion starting from $(1, 0, 0)$. Denoting by $|\cdot|$ the Euclidean norm in $\mathbb{R}^3$, the process

$$X_t := \frac{1}{|W_t|}, \quad t \geq 0,$$

defines a strictly positive local martingale in the filtration $(\mathcal{F}_t)$ generated by W with dynamics

$$dX_t = -X_t^2 dB_t,$$

where B denotes a one-dimensional $(\mathcal{F}_t)$-Brownian motion. Explicit calculations show that $t \mapsto \mathbb{E}X_t$ is a strictly decreasing function converging towards zero for $t \to \infty$.

In [24] it is shown that the strictness of a continuous positive local martingale can be measured in terms of the tail behaviour of the distribution of its supremum respectively its quadratic variation.

Theorem 1.1.4. *Let M be a non-negative continuous local martingale with $\mathbb{E}M_0 < \infty$. Then*

$$\mathbb{E}^{\mathsf{P}}(M_0 - M_\infty) = \lim_{x\to\infty} x \cdot \mathsf{P}\left(\sup_{t\geq 0} M_t \geq x\right) = \sqrt{\frac{\pi}{2}} \cdot \lim_{x\to\infty} x \cdot \mathsf{P}\left(\langle M\rangle_\infty^{1/2} \geq x\right).$$

An extension of this result to locally square-integrable local martingales with bounded jumps can be found in [50]. Note that the above theorem implies that a continuous positive strict local martingale has to be very volatile. This is further confirmed by the following result from [20].

Theorem 1.1.5. *Let X be defined by the SDE $dX_t = \sigma(X_t)dW_t$ for some Brownian motion W with $X_0 = 1$ and let us suppose that $\sigma(\cdot)$ is bounded and bounded away from zero on compact sets of $(0, \infty)$ with $\sigma(0) = 0$. Then X is a strict local martingale if and only if*

$$\int_1^\infty \frac{x}{\sigma^2(x)} dx < \infty.$$

The class of local martingales plays an important role in stochastic analysis since it is stable with respect to stochastic integration in the following sense: Given a local martingale M, for any locally bounded predictable process h the stochastic integral $\int hdM$ is again a local martingale. Moreover, Section 1.2.3 below hints at the significance of strict local martingales in mathematical finance.

1.1.2 The usual assumptions

Definition 1.1.6. A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ is said to satisfy the usual assumptions if

i) $\mathcal{F}_0$ contains all the P-negligible sets of $\mathcal{F}$, i.e. all $A \subset \Omega$ for which there exists $B \in \mathcal{F}$ such that $A \subset B$ and $\mathsf{P}(B) = 0$;

ii) $\mathcal{F}_t = \cap_{u>t} \mathcal{F}_u$ for all $t \geq 0$, i.e. $(\mathcal{F}_t)$ is right-continuous.

It is very convenient to suppose that a probability space satisfies the usual assumptions. For example, in this case every martingale has a càdlàg version, cf. Theorem 2.9 in [67]. However, sometimes the usual assumptions are too restrictive. This is e.g. the case if one wants to construct a new probability measure which is not absolutely continuous with respect to P. We will encounter this situation in Subsection 2.2.2 of this thesis, when we construct a new probability measure via a strict local martingale. In Chapter 3 we therefore introduce a new kind of augmentation of filtrations suitable for such kind of measure changes.

1.1.3 Enlargement of filtrations

The study of enlargements of filtrations started with works by Barlow, Jeulin, and Yor in the late seventies, cf. [7, 42, 43, 44]. In all generality the problem of enlargements of filtrations is the following: Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ is a filtered probability space satisfying the usual assumptions and assume that $(\mathcal{D}_t)$ is another filtration of $\mathcal{F}$. Then the enlarged filtration $(\mathcal{G}_t)$ defined by

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{D}_s)$$

is the smallest filtration satisfying the usual assumptions such that each $\mathcal{G}_t$ contains $\mathcal{F}_t$ and $\mathcal{D}_t$ for all $t \geq 0$. If $\mathcal{D}_t = \mathcal{D}_0$ for all $t \geq 0$, then we have an initial enlargement of filtration. If the $\mathcal{D}_t$ are increasing over time, the operation is known as a progressive enlargement of filtration.

The principal question in the theory of enlargement of filtrations is the following: Does an $(\mathcal{F}_t)$-martingale remain a $(\mathcal{G}_t)$-semimartingale? And if this is the case, what is its semimartingale decomposition in the enlarged filtration $(\mathcal{G}_t)$? In this context the following two hypotheses were introduced in the literature:

$(\mathcal{H})$ Every $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-martingale.

$(\mathcal{H}')$ Every $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-semimartingale.

While the $(\mathcal{H})$-hypothesis is very restrictive but well-understood, the $(\mathcal{H}')$-hypothesis has so far been intensively studied in the following two cases only, which are special cases of either the initial or the progressive enlargement setting:

- $\mathcal{D}_t = \sigma(L)$ for some $\mathcal{F}$-measurable random variable L for all $t \geq 0$.
- $\mathcal{D}_t = \sigma(\mathbb{1}_{\{\sigma \leq s\}}; s \leq t)$ for some positive $\mathcal{F}$-measurable random variable σ, also called a random time.

There are very few studies done outside of these two specific frameworks, for a recent one cf. [47].

In Chapter 4 of this thesis we are interested in progressive enlargements of filtrations with a random time σ as defined above. Hence, we will concentrate on this type of enlargement for the rest of this subsection. Therefore, henceforth we assume that $\mathcal{D}_t = \sigma(\mathbb{1}_{\{\sigma \leq s\}}; s \leq t)$. We will need the notion of the dual predictable projection defined in the following theorem.

Theorem 1.1.7. *Let C be an integrable right-continuous increasing process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ satisfying the usual assumptions. Then there exists a predictable increasing process A which is unique up to P-indistinguishability such that*

$$\mathbb{E}\left(\int_0^\infty Y_s dC_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s\right)$$

for any positive $(\mathcal{F}_t)$-predictable process Y. The process A is called the dual predictable projection of C.

Suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ satisfies the usual assumptions. An important role is played by the so called Azéma supermartingale $Z_t := \mathsf{P}(\sigma > t | \mathcal{F}_t)$, which we may assume to be càdlàg. The Doob-Meyer decomposition of this supermartingale is

$$Z_t = m_t - A_t$$

with m being an $(\mathcal{F}_t)$-martingale and A denoting the dual predictable projection of $(\mathbb{1}_{\{\sigma \leq t\}})_{t \geq 0}$ with respect to $(\mathcal{F}_t)$. If σ avoids stopping times, i.e. $\mathsf{P}(\sigma = \tau) = 0$ for all $(\mathcal{F}_t)$-stopping times τ, then A is continuous and we have $m_t = \mathbb{E}(A_\infty | \mathcal{F}_t)$, which is a BMO martingale, cf. [22]. Concerning the $(\mathcal{H}')$-hypothesis, it is shown in all generality in [43] that any $(\mathcal{F}_t)$-martingale stopped at σ is also a $(\mathcal{G}_t)$-semimartingale and its $(\mathcal{G}_t)$-decomposition is derived. For our purposes the following theorem suffices.

Theorem 1.1.8. *Assume that σ avoids stopping times. Any local $(\mathcal{F}_t)$-martingale M stopped at σ is a $(\mathcal{G}_t)$-semimartingale with decomposition*

$$M_{t \wedge \sigma} = \widetilde{M}_t + \int_0^{t \wedge \sigma} \frac{d\langle m, M \rangle_s}{Z_{s-}}$$

for some local $(\mathcal{G}_t)$-martingale $\widetilde{M}$.

In general the semimartingale property in the enlarged filtration may get lost after time σ. However, for a certain class of random times called honest times, which are ends of optional sets, every $(\mathcal{F}_t)$-martingale remains a $(\mathcal{G}_t)$-semimartingale on the whole time horizon and thus the $(\mathcal{H}')$-hypothesis holds.

1.1.4 Girsanov theorem

The following result describes how martingales behave with respect to an equivalent change of probability measure. Part ii) of Theorem 1.1.9 is known as Girsanov's theorem.

Theorem 1.1.9. *Let $\rho = (\rho_t)$ be a càdlàg strictly positive uniformly integrable martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ with $\mathbb{E}^{\mathsf{P}}\rho_\infty = 1$. Then $\mathsf{Q} = \rho_\infty.\mathsf{P}$ defines a probability measure, which is equivalent to P, with $\mathsf{Q}(A) = \mathbb{E}^{\mathsf{P}}(\rho_\infty \mathbb{1}_A)$, $A \in \mathcal{F}$. Let $M = (M_t)_{t\geq 0}$ be an adapted càdlàg process.*

i) M is a local Q-martingale if and only if $M\rho$ is a local P-martingale.

ii) If M is a local P-martingale, then

$$M_t - \int_0^t \frac{d\,[M,\rho]_s}{\rho_s}, \quad t \geq 0,$$

is a local Q-martingale.

Conversely, if we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ and an equivalent probability measure $\mathsf{Q} \sim \mathsf{P}$, then the Radon-Nikodym density process defined as

$$\rho_t := \left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_{\mathcal{F}_t}$$

is a uniformly integrable martingale. If $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ satisfies the usual assumptions, then we can choose a càdlàg version of $\rho = (\rho_t)$ and the above theorem applies.

In this thesis we study changes of probability measure which are out of the scope of Theorem 1.1.9. On the one hand, in Chapter 2 and Chapter 3 we are interested in the case when $\rho = (\rho_t)$ is no longer a uniformly integrable P-martingale, but only a local martingale. On the other hand, in Chapter 4 we study changes of probability measure up to random times which in general fail to be $(\mathcal{F}_t)$-stopping times. Since a uniformly integrable $(\mathcal{F}_t)$-martingale stopped at an arbitrary random time may lose its martingale property, the situation is more delicate in this case.

1.2 Financial markets

In this section we give a short introduction to no arbitrage theory and option pricing in continuous time. Throughout we suppose that the interest rate of the riskless bond is normalized to zero, that there are no frictions like transaction costs in the market, and that the stocks do not pay any dividends.

1.2.1 First fundamental theorem of asset pricing

The most general version of the first fundamental theorem of asset pricing in continuous time states the equivalence between a certain no arbitrage condition called No Free Lunch with Vanishing Risk fulfilled by an $\mathbb{R}^d$-valued semimartingale and the existence of an equivalent sigma martingale measure for this process, cf. [17] and the textbook [18].

Given an $\mathbb{R}^d$-valued semimartingale S on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ we call an $\mathbb{R}^d$-valued predictable process H an admissible strategy for S if the stochastic integral $\int H_t dS_t$ is well-defined and there exists a constant $a \in \mathbb{R}_+$ such that

$$\int_0^t H_u dS_u \geq -a \quad \text{a.s. for all } t \geq 0.$$

The set

$$K = \left\{ V_\infty \;\middle|\; H \text{ is admissible and } V_\infty := \lim_{t\to\infty} \int_0^t H_u dS_u \text{ exists a.s.} \right\}$$

consists of all terminal portfolio values corresponding to admissible strategies for S starting from zero initial investment. Moreover, the set

$$C = \{g \in L^\infty(\mathsf{P}) \mid \exists\, f \in K \text{ s.t. } g \le f\}$$

describes all essentially bounded claims which can be superhedged with zero cost.

Definition 1.2.1. We say that the semimartingale S satisfies the condition of

i) no arbitrage (NA) if $C \cap L^\infty_+ = \{0\}$,

ii) no free lunch with vanishing risk (NFLVR) if $\overline{C} \cap L^\infty_+ = \{0\}$, where $\overline{C}$ denotes the closure of C in the L^∞-topology.

In this thesis we only deal with non-negative stock price processes. Since a non-negative sigma martingale is a local martingale, we will use the following version of the first fundamental theorem of asset pricing:

Theorem 1.2.2. *The following assertions are equivalent for an $\mathbb{R}^d_+$-valued semimartingale model $S = (S_t)_{t\ge 0}$ of a financial market:*

1. *There is a probability measure P^* equivalent to P such that S is a local martingale under P^*.*

2. *S satisfies the condition of no free lunch with vanishing risk.*

Once we know that the market satisfies NFLVR, we can introduce derivatives and price them in an arbitrage-free way. For this let us fix some equivalent local martingale measure P^*. Then an arbitrage-free price process for a European option with payoff $R \in \mathcal{F}_T$ at maturity T is given by

$$R_t := \mathbb{E}^{\mathsf{P}^*}(R|\mathcal{F}_t), \quad t \le T.$$

As long as all derivatives in the market are priced at time t according to their conditional expectation with respect to $\mathcal{F}_t$ and the same measure P^*, the NFLVR condition will still be satisfied after the introduction of these new financial products.

1.2.2 Option prices as probabilities

The most famous formula in quantitative finance, which was established by Black, Scholes, and Merton in the seventies, computes the price of a call option written on a stock whose price process S is a geometric Brownian motion with initial value $s > 0$ and volatility $\upsilon > 0$. The Black-Scholes formula states that the price of a call with strike $K > 0$ and maturity $T > 0$ equals

$$C(K,T) := \mathbb{E}^{\mathsf{P}^*}(S_T - K)^+ = s \cdot \Phi\left(\frac{\ln(s/K)}{\upsilon\sqrt{T}} + \frac{\upsilon\sqrt{T}}{2}\right) - K \cdot \Phi\left(\frac{\ln(s/K)}{\upsilon\sqrt{T}} - \frac{\upsilon\sqrt{T}}{2}\right),$$

where Φ denotes the standard normal cumulative distribution function and P^* the risk-neutral probability measure. Similarly, the price of a put option with strike $K > 0$ and maturity $T > 0$, taking $s = \upsilon = 1$, equals

$$P(K,T) := \mathbb{E}^{\mathsf{P}^*}(K - S_T)^+ = K \cdot \Phi\left(\frac{\ln(K)}{\sqrt{T}} + \frac{\sqrt{T}}{2}\right) - \Phi\left(\frac{\ln(K)}{\sqrt{T}} - \frac{\sqrt{T}}{2}\right).$$

The measure P^* is a priori only defined on the finite time interval $[0, T]$ for every $T > 0$, but let us suppose that we can extend it to the infinite time horizon such that for all $t \geq 0$,

$$S_t = \exp\left(W_t^* - \frac{t}{2}\right)$$

for some P^*-Brownian motion W^*. In [52] the following remarkable identity is shown:

$$P(K, T) = K \cdot \mathsf{P}^*(g_K \leq T),$$

where g_K denotes the last passage time of S at level K, i.e.

$$g_K := \sup\{t \geq 0 : \; S_t = K\},$$

where we set $\sup \emptyset := 0$. Note that $\mathsf{P}^*(g_K < \infty) = 1$, since S_t converges P^*-almost surely to zero as $t \to \infty$.

In fact, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ this result can be extended to any non-negative càdlàg local $(\mathcal{F}_t)$-martingale M with $M_0 = 1$ that converges to zero almost surely and whose supremum process is continuous. In this case,

$$\mathbb{E}(K - M_T)^+ = K \cdot \mathsf{P}(g_K^M \leq T), \quad \text{where } g_K^M := \sup\{t \geq 0 : \; M_t = K\}. \tag{1.1}$$

Assuming that M is a true martingale and that there exists a probability measure P^M such that for all $t \geq 0$,

$$\mathsf{P}^M\big|_{\mathcal{F}_t} = M_t \cdot \mathsf{P}|_{\mathcal{F}_t},$$

a similar formula can be derived for the call option. In this case one has for any $T > 0$,

$$\mathbb{E}(M_T - K)^+ = \mathsf{P}^M(g_K^M \leq T). \tag{1.2}$$

However, if M is a strict local martingale the situation is more delicate and we will discuss this case in Section 2.6.
Note that the representations (1.1) and (1.2) of the prices of the call resp. put option are model independent and can hence be seen as generalized Black-Scholes formulae. More on this topic can be found in the monograph [64].

1.2.3 Bubbles

Let $S = (S_t)_{t \geq 0}$ denote the positive price process of a stock traded on the time interval $[0, T]$ for some $T \in \mathbb{R}_+$. If the market participants are rational, the price S_t of the stock at time t should agree with its fundamental value at that time denoted by S_t^*. If the difference between those two values is not equal to zero, then from an economic point of view an asset price bubble exists and its size equals

$$\beta_t := S_t - S_t^*.$$

Assuming the existence of an equivalent local martingale measure P^* for S, the fundamental value is given by $S_t^* = \mathbb{E}^{\mathsf{P}^*}(S_T | \mathcal{F}_t)$. Therefore, a bubble exists if and only if S is a strict local P^*-martingale. Especially note that $\beta_t \geq 0$ almost surely for all $t \leq T$. Moreover, put-call parity does not hold at time $t < T$ if $\beta_t > 0$. Indeed, in this case for any $K > 0$,

$$\underbrace{\mathbb{E}^{\mathsf{P}^*}\left((S_T - K)^+ | \mathcal{F}_t\right)}_{\text{Call price at time } t} - \underbrace{\mathbb{E}^{\mathsf{P}^*}\left((K - S_T)^+ | \mathcal{F}_t\right)}_{\text{Put price at time } t} = S_t^* - K < S_t - K.$$

In Chapter 2 we study the influence of asset price bubbles on the pricing of derivatives in greater generality. More information on the plausibility of strict local martingales in mathematical finance can be found in [32] and Section 5 of [66].

1.2.4 Insider trading and credit default risk models

The theory of enlargement of filtrations has two main application areas in mathematical finance. On the one hand it provides a natural setup for the modelling of insider information. Insider knowledge via an initial enlargement of filtration has e.g. been studied in [4] and [62], while in [33] free lunch opportunities for an insider at a random time are analyzed in a progressively enlarged filtration. On the other hand, progressive enlargements of filtrations with random times have been successfully employed to study credit default risk via the reduced form approach originally developed in [23]. In general there are two approaches in the literature to credit risk modeling: the structural approach and the reduced form approach. While in the structural approach the firm's value process is directly modeled and the default time is defined as the first crossing time of the firm's value process through some barrier and is hence predictable, the reduced form approach allows to model credit default times in an unpredictable way. The advantage of the latter one is that credit spreads do in general not converge to zero when maturity goes to zero, a phenomenon observed in real-world financial markets.
Therefore, an interesting question for financial applications is, if some kind of no arbitrage condition is preserved if one enlarges the filtration with a random time σ. In Section 4.6 we provide some answer to this question on the time horizon $[0, \sigma]$ assuming that the original market model satisfies NFLVR.

1.3 Overview of Chapters 2, 3, and 4

Each of the next three chapters contains a different project and starts with its own introduction. In the following we briefly point out what the projects are about.
In Chapter 2 we discuss càdlàg strict local martingales and their role in the modelling of asset price bubbles. First we associate with any càdlàg strictly positive strict local martingale a new probability measure, under which its reciprocal becomes a true martingale. With respect to this measure we then express the risk-neutral price of an option in a way which shows the influence of the asset price bubble of the underlying on the valuation of the option. We treat certain classes of European path-dependent options as well as American and European exchange options. In the latter case the option value can be expressed using last passage times. This result is also applied to the "real-world" pricing of standard European and American call options. Finally, we consider continuous multivariate (strict) local martingales and look whether we can find a similar change of measure as in the one-dimensional case. Afterwards, in Chapter 3 a new kind of augmentation of filtrations suitable for a change of probability measure by a strict local martingale is introduced.
In Chapter 4 we discuss changes of probability measure up to random times. First we give explicit examples of such measure changes for the classes of honest times and pseudo-stopping times (for a definition see Chapter 4). Then we discuss the question of no arbitrage up to a random time and we derive a sufficient condition in terms of the multiplicative decomposition of the Azéma supermartingale for NFLVR to hold true on a random time horizon. We end by proving an extension of Girsanov's theorem after honest times and we point out the relation to a class of processes called relative martingales.

Chapter 2 contains a preliminary version of [46] which is joint work with Constantinos Kardaras from the London School of Economics and my advisor Ashkan Nikeghbali from the University of Zurich. Chapter 3 is a slightly modified version of [49] which is joint work with Ashkan Nikeghbali and Chapter 4 is based on the single-authored paper [48].

Chapter 2

Strict local martingales and bubbles

In this chapter we analyze asset price bubbles modeled by strict local martingales. With any strict local martingale one can associate a new measure, which is studied in detail in the first part of the chapter. In the second part we determine the "default term" apparent in risk-neutral option prices if the underlying stock exhibits a bubble modeled by a strict local martingale. Results for certain path dependent options and last passage time formulas are given.

2.1 Introduction

The goal of this chapter is to determine the influence of asset price bubbles on the pricing of derivatives. Asset price bubbles have been studied extensively in the economic literature looking for explanations of why they arise, but have only recently gained attention in mathematical finance by Cox and Hobson (2005), Pal and Protter (2010), and Jarrow et al. (2007, 2009, 2010). When an asset price bubble exists, the market price of the asset is higher than its fundamental value. From a mathematical point of view this is the case, when the stock price process is modeled by a positive strict local martingale under the equivalent local martingale measure. Here by a strict local martingale we understand a local martingale, which is not a true martingale. Strict local martingales were first studied in the context of financial mathematics by Delbaen and Schachermayer (1995). Afterwards Elworthy et al. (1997, 1999) studied some of their properties including their tail behaviour. More recently, the interest in them grew again (cf. e.g. Mijatovic and Urusov (2012)) because of their importance in the modelling of financial bubbles.
Obviously, there are options for which well-known results regarding their valuation in an arbitrage-free market hold true without modification, regardless of whether the underlying is a strict local martingale or a true martingale under the risk-neutral measure. One example is the put option with strike $K \geq 0$. If the underlying is modeled by a continuous local martingale X with $X_0 = 1$, it is shown by Madan et al. (2008b) that the risk-neutral value of the put option can be expressed in terms of the last passage time of the local martingale X at level K via

$$\mathbb{E}(K - X_T)^+ = \mathbb{E}\left((K - X_\infty)^+ \mathbb{1}_{\{\rho_K^X \leq T\}}\right) \quad \text{with} \quad \rho_K^X = \sup\{t \geq 0 | X_t = K\}.$$

This formula does not require X to be a true martingale, but is also valid for strict local martingales. However if we go from puts to calls, the strict locality of X is relevant. The general idea is to reduce the call case to the put case by a change of measure with Radon-Nikodym density process given by $(X_t)_{t\geq 0}$ as done in Madan et al. (2008b) in the case where X is a true martingale. However, if X is a strict local martingale, this does not define a measure any more. Instead, we first have to localize the strict local martingale and can thus only define measures on stopped sub-σ-algebras. Under certain conditions on the probability space, we can then extend the so-defined consistent family of measures to a measure defined on some larger σ-field. Under the new measure the reciprocal of X turns into a true martingale. The conditions we impose are taken from Föllmer (1972), who requires the filtration to be a standard system (cf. Definition 2.2.5). This way we get an extension of Theorem 4 in Delbaen and Schachermayer (1995) to general probability spaces and càdlàg local martingales. We study the behavior of X and other local martingales under the new measure.

Using these technical results we obtain decomposition formulas for some classes of European path-dependent options under the NFLVR condition. These formulas are extensions of Proposition 2 in Pal and Protter (2010), which deals with non-path-dependent options. We decompose the option value into a difference of two positive terms, of which the second one shows the influence of the stock price bubble.

Furthermore, we express the risk-neutral price of an exchange option in the presence of asset price bubbles as an expectation involving the last passage time at the strike level under the new measure. This result is similar to the formula for call options derived by Madan, Roynette and Yor (2008a) or Yen and Yor (2009) for the case of reciprocal Bessel processes. We can further generalize their formula to the case where the candidate density process for the risk-neutral measure is only a strict local martingale. Then the NFLVR condition is not fulfilled and risk-neutral valuation fails, so that we have to work under the real-world measure. Since in this case the price of a zero coupon bond is decreasing in maturity even with an interest rate of zero, some people refer to this as a bond price bubble as opposed to the stock price bubbles discussed above, cf. e.g. Hulley (2010). In this general setup we obtain expressions for the option value of European and American call options in terms of the last passage time and the explosion time of the deflated price process, which make some anomalies of the prices of call options in the presence of bubbles evident: European calls are not increasing in maturity any longer and the American call option premium is not equal to zero any more, cf. Cox and Hobson (2005).

This chapter is organized as follows: In the next section we study strictly positive (strict) local martingales in more detail. On the one hand, we demonstrate ways of how one can obtain strict local martingales, while on the other hand we construct the above mentioned measure associated with a càdlàg strictly positive local martingale on a general filtered probability space with a standard system as filtration. We give some examples of this construction in Section 2.3. In Section 2.4 we then apply our results to the study of asset price bubbles. After formally defining the financial market model we obtain decomposition formulas for certain classes of European path-dependent options, which show the influence of stock price bubbles on the value of the options under the NFLVR condition. In Section 2.5 we further study the relationship between the original and the new measure constructed in Section 2.2.2, which we apply in Section 2.6 to obtain last passage time formulas for the European and American exchange option in the presence of asset price bubbles. Moreover, we show how this result can be applied to the real-world pricing of European and American call options. The last section contains some results about multivariate strict local martingales.

2.2 Càdlàg strictly positive strict local martingales

When dealing with continuous strictly positive strict local martingales a very useful tool is the result from [16], see also Proposition 6 in [60], which states that every such process defined as the coordinate process on the canonical space of trajectories can be obtained as the reciprocal of a "Doob h-transform"[1] with $h(x) = x$ of a continuous non-negative true martingale. Conversely, any such transformation of a continuous non-negative martingale, which hits zero with positive probability, yields a strict local martingale.
The goal of this section is to extend these results to càdlàg processes and general probability spaces satisfying some extra conditions, which were introduced in [61] and used in a similar context in [29]. While the construction of strict local martingales from true martingales follows from an application of the Lenglart-Girsanov theorem, the converse theorem relies as in [16] on the construction of the Föllmer exit measure of a strictly positive local martingale as done in [29] and [54].

2.2.1 How to obtain strictly positive strict local martingales

Examples of continuous strict local martingales have been known for a long time, the canonical example being the reciprocal of a Bessel process of dimension 3. This example can be generalized to a broader class of transient diffusions, which taken in natural scale turn out to be strict local martingales, cf. [24]. A natural way to construct strictly positive continuous strict local martingales is given in Theorem 1 of [16]. There, it is shown that every uniformly integrable non-negative martingale with positive probability to hit zero gives rise to a change of measure such that its reciprocal is a strict local martingale under the new measure. For the non-continuous case and for not necessarily uniformly integrable martingales we now give a simple extension of the just mentioned theorem from [16]:

Theorem 2.2.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{Q})$ be the natural augmentation of some filtered probability space with $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, i.e. the filtration $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}_t$-negligible sets for all $t \geq 0$. Let Y be a non-negative Q-martingale starting from $Y_0 = 1$. Set $\tau = \inf\{t \geq 0 : Y_t = 0\}$ and assume that $\mathsf{Q}(\tau < \infty) > 0$. Furthermore, suppose that Y does not jump to zero Q-almost surely. For all $t \geq 0$, define a probability measure P_t on $\mathcal{F}_t$ via $\mathsf{P}_t = Y_t.\mathsf{Q}|_{\mathcal{F}_t}$; in particular, $\mathsf{P}_t \ll \mathsf{Q}|_{\mathcal{F}_t}$. Assume that either Y is uniformly integrable under Q or that the non-augmented probability space satisfies condition (P)*[2]*. Then, we can extend the consistent family $(\mathsf{P}_t)_{t\geq 0}$ to a measure P on the augmented space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$. Under the measure P the process Y does never reach zero and its reciprocal $1/Y$ is a strict local P-martingale.*

Proof. Since the underlying probability space satisfies the natural assumptions, we may choose a càdlàg version of Y, cf. Propositions 3.1 and 3.3 in [57]. Especially, this means that τ is a well-defined stopping time. If Y is a uniformly integrable martingale, the measure P is defined on $\mathcal{F}$ by $d\mathsf{P} = Y_\infty d\mathsf{Q}$. In the other case, when the probability space fulfills condition (P), the existence of the measure P follows from Corollary 4.9 in [57]. Moreover note that

$$\mathsf{P}(\tau < \infty) = \lim_{t\to\infty} \mathsf{P}(\tau \leq t) = \lim_{t\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau\leq t\}} Y_t\right) = 0,$$

therefore the process $1/Y$ is a P-almost surely well defined semimartingale. The result now follows from Corollary 3.10 in Chapter III of [36] applied to $M'_t := \frac{1}{Y_t}\mathbb{1}_{\{\tau>t\}}$, once we

[1]Note that we abuse the word "Doob h-transform" in this context slightly, since Doob h-transforms are normally only defined in the theory of Markov processes.

[2]Condition (P) first appeared in [61] and was later used in [57]. We recall its definition in Appendix 2.7.

can show that $(M'_{t\wedge\tau_n} Y_{t\wedge\tau_n})$ with $\tau_n = \inf\{t \geq 0 : \ Y_t \leq \frac{1}{n}\}$ is a local Q-martingale for every $n \in \mathbb{N}$. But,

$$M'_{t\wedge\tau_n} Y_{t\wedge\tau_n} = \mathbb{1}_{\{\tau > t\wedge\tau_n\}} = 1 \quad \mathsf{Q}\text{-a.s.}$$

because Y does not jump to zero Q-almost surely. This trivially proves the martingale property. Finally, the strictness of the local martingale $1/Y$ under P follows from

$$\mathbb{E}^{\mathsf{P}}\left(\frac{1}{Y_t}\right) = \mathsf{Q}(\tau > t) < 1$$

for t large enough, since by assumption $\mathsf{Q}\,(\tau < \infty) > 0$. □

Starting with a Brownian motion stopped at zero under Q, it is easy to show that the associated strict local martingale under P is the reciprocal of the three-dimensional Bessel process, which is the canonical example of a strict local martingale (cf. Example 1 in [60]). Without stating the general result, the above construction is also applied in [12] to construct examples of strict local martingales with jumps related to Dunkl Markov processes on the one hand (cf. Proposition 3 in [12]) and semistable Markov processes on the other hand (cf. Proposition 5 in [12]). Apart from the previous, there do not seem to be any well-known examples of strict local martingales with jumps. Note, however, that one can construct an example by taking any continuous strict local martingale and multiplying it with the stochastic exponential of an independent compound Poisson process or any other independent and strictly positive jump martingale.

In the following example we construct a "non-trivial" positive strict local martingale with jumps by a shrinkage of filtration.

Example 2.2.2. Consider the well-known reciprocal three-dimensional Bessel process Y as a function of a three-dimensional standard Brownian motion $B = (B^1, B^2, B^3)$ starting from $B_0 = (1, 0, 0)$, i.e.

$$Y = \frac{1}{\sqrt{(B^1)^2 + (B^2)^2 + (B^3)^2}}.$$

We define the filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ through $\mathcal{F}_t = \sigma(B^1_s, B^2_s, B^3_s; s \leq t)$ and $\mathcal{G}_t = \sigma(B^1_s, B^2_s; s \leq t)$, as well as the filtration $(\mathcal{H}_t)_{t\geq 0}$ through

$$\mathcal{H}_t = \mathcal{F}_{\frac{\lfloor nt\rfloor}{n}} \vee \mathcal{G}_t = \sigma\left(B^1_s, B^2_s, s \leq t;\ B^3_u, u \leq \frac{\lfloor nt\rfloor}{n}\right)$$

for some $n \in \mathbb{N}$. It is shown in Theorem 15 of [30] that not only Y itself is a strict local $(\mathcal{F}_t)_{t\geq 0}$-martingale, but that also the optional projection of Y onto $(\mathcal{G}_t)_{t\geq 0}$ is a continuous local $(\mathcal{G}_t)_{t\geq 0}$-martingale. Since $\mathcal{G}_t \subset \mathcal{H}_t \subset \mathcal{F}_t$ for $t \geq 0$, it follows by Corollary 2 of [30] that then the optional projection of Y onto $(\mathcal{H}_t)_{t\geq 0}$, denoted by ${}^\circ Y$, is also a local martingale. However, since its expectation process is decreasing, ${}^\circ Y$ must be a strict local martingale that jumps at $t \in \frac{\mathbb{N}}{n}$. Indeed, since B^3 is a Brownian motion independent of B^1 and B^2, B^3_t given $\mathcal{H}_t$ is normally distributed with mean $B^3_{\frac{\lfloor nt\rfloor}{n}}$ and variance $t - \frac{\lfloor nt\rfloor}{n}$. Therefore, ${}^\circ Y$ is given by the explicit formula ${}^\circ Y_t = u(B^1_t, B^2_t, B^3_{\frac{\lfloor nt\rfloor}{n}}, t)$, where

$$u(x, y, a, t) = \int_{\mathbb{R}} (x^2 + y^2 + z^2)^{-1/2} \cdot \sqrt{\frac{1}{2\pi\left(t - \frac{\lfloor nt\rfloor}{n}\right)}} \exp\left(-\frac{1}{2\left(t - \frac{\lfloor nt\rfloor}{n}\right)}(z - a)^2\right) dz.$$

Remark 2.2.3. In the recent preprint [66] the method of filtration shrinkage is applied in greater generality to construct more sophisticated examples of strict local martingales with jumps.

Example 2.2.4. As a further example, any nonnegative non-uniformly integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale Z with $Z_0 = 1$ allows to construct a strictly positive strict local martingale Y relative to a new filtration $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ through a deterministic change of time: simply set

$$Y_t = \begin{cases} \frac{1}{2}\left(1 + Z_{\frac{t}{1-t}}\right) & : \ 0 \leq t < 1 \\ \frac{1}{2}\left(1 + \lim_{t\to\infty} Z_t\right) & : \ 1 \leq t \end{cases}$$

and define $\widetilde{\mathcal{F}}_t = \mathcal{F}_{\frac{t}{1-t}}$ for $t < 1$ and $\widetilde{\mathcal{F}}_t = \mathcal{F}_\infty$ for $t \geq 1$. Since Z is *not* uniformly integrable, we have $\mathbb{E}Y_1 < Y_0 = Z_0 = 1$ almost surely. Note however that Y is a true martingale on the interval $[0, 1)$. Instead of setting Y constant for $t \geq 1$ one can also define Y to behave like any other strictly positive local martingale starting from $Y_1 := \frac{1}{2}(1 + \lim_{t\to\infty} Z_t)$ on $[1, \infty)$.

2.2.2 From strictly positive strict local martingales to true martingales

In the following let $(\Omega, \mathcal{F}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space. Furthermore, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the right-continuous augmentation of $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$, i.e. $\mathcal{F}_t := \widetilde{\mathcal{F}}_{t+} = \bigcap_{s>t} \widetilde{\mathcal{F}}_s$ for all $t \geq 0$. Note, however, that the filtration is *not* completed with the negligible sets of $\mathcal{F}$.

Definition 2.2.5. (cf. [29]) Let T be a partially ordered non-void index set and let $(\widetilde{\mathcal{F}}_t)_{t\in T}$ be a filtration on Ω. Then $(\widetilde{\mathcal{F}}_t)_{t\in T}$ is called a standard system if

- each measurable space $(\Omega, \widetilde{\mathcal{F}}_t)$ is a standard Borel space, i.e. $\widetilde{\mathcal{F}}_t$ is σ-isomorphic to the σ-field of Borel sets on some complete separable metric space.
- for any increasing sequence $(t_i)_{i\in\mathbb{N}} \subset T$ and for any $A_1 \supset A_2 \supset \cdots \supset A_i \supset \ldots$, where A_i is an atom of $\widetilde{\mathcal{F}}_{t_i}$, we have $\bigcap_i A_i \neq \emptyset$.

As noted in [57] the filtration $\widetilde{\mathcal{F}}_t = \sigma(X_s, s \leq t)$, where $X_t(\omega) = \omega(t)$ is the coordinate process on the space $C(\mathbb{R}_+, \mathbb{R}_+)$ of non-explosive non-negative continuous functions, is not a standard system. However, it will be seen below that when dealing with strict local martingales it is natural to work on the space of all $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$-valued processes that are continuous up to some time $\alpha \in [0, \infty]$ and constant afterwards. As noted in example (6.3) in [29] the filtration generated by the coordinate process on this space is indeed a standard system. More generally, we have the following lemma.

Lemma 2.2.6. *Let $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$ be the space of functions from $\mathbb{R}_+$ into $\overline{\mathbb{R}}^n_+$ with componentwise right-continuous paths $(\omega_i(t))_{t\geq 0}$, $i = 1, \ldots, n$, that have left limits on $(0, \alpha(\omega))$ for some $\alpha(\omega) \in [0, \infty]$ and remain constant on $[\alpha(\omega), \infty)$ at the value $\lim_{t\uparrow\alpha(\omega)} \omega_i(t)$ if this limit exists and at ∞ otherwise. We denote by $(X_t)_{t\geq 0}$ the coordinate process, i.e. $X_t(\omega_1, \ldots, \omega_n) = (\omega_1(t), \ldots, \omega_n(t))$, and by $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ the canonical filtration generated by the coordinate process, i.e. $\widetilde{\mathcal{F}}_t = \sigma(X_s;\ s \leq t)$. Furthermore, set $\mathcal{F} = \bigvee_{t\geq 0} \widetilde{\mathcal{F}}_t$. Then, $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is a standard system on the space $(\Omega, \mathcal{F}, (\widetilde{\mathcal{F}}_t)_{t\geq 0})$. The same is true, if we replace $D'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$ by its subspace $C'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$ of functions which are componentwise continuous on some $(0, \alpha(\omega))$ and remain constant on $[\alpha(\omega), \infty)$ at the value $\lim_{t\uparrow\alpha(\omega)} \omega_i(t)$ if this limit exists and at ∞ otherwise.*

Proof. We prove the claim for $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$. The case $\Omega = C'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$ is done in a similar way. As in [21] we define a bijective mapping i from Ω to some subspace $A \subset (\overline{\mathbb{R}}^n_+)^{\mathbb{Q}}$, (where here $\mathbb{Q}$ denotes the set of all rational numbers), via $\omega \mapsto (X_r(\omega))_{r\in\mathbb{Q}}$. It is clear that i is bijective and we have $\mathcal{F} = i^{-1}(\mathcal{B}(A))$. Furthermore, a sequence $A_1 \supset A_2 \supset \cdots \supset A_i \supset \ldots$ of atoms of $\mathcal{F}_{t_i} = \sigma(X_s;\ s \le t_i)$ defines a component-wise càdlàg function on the interval $[0, \lim t_i] \cap [0, \alpha(\omega))$, which is constant on $[0, \lim t_i] \cap [\alpha(\omega), \infty)$, for every increasing sequence $(t_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$. This function can easily be extended to an element of $D'(\mathbb{R}_+, \overline{\mathbb{R}}^n_+)$. $\square$

Recall that for any $(\mathcal{F}_t)_{t\ge0}$-stopping time τ the sigma-algebra $\mathcal{F}_{\tau-}$ is defined as

$$\mathcal{F}_{\tau-} = \sigma(\widetilde{\mathcal{F}}_0, \{\{\tau > t\} \cap \Gamma :\ \Gamma \in \mathcal{F}_t,\ t > 0\}).$$

Lemma 2.2.7. (cf. [29], Remark 6.1) *Let $(\widetilde{\mathcal{F}}_t)_{t\ge0}$ be a standard system on Ω. Then for any increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $(\mathcal{F}_t)$-stopping times the family $(\mathcal{F}_{\tau_n-})_{n\in\mathbb{N}}$ is also a standard system.*

Notation: When working on the subspace $(\Omega, \mathcal{F}_{\tau-})$ of $(\Omega, \mathcal{F})$, where τ is some $(\mathcal{F}_t)$-stopping time, we must restrict the filtration to $(\mathcal{F}_{t\wedge\tau-})_{t\ge0}$, where with a slight abuse of notation we set $\mathcal{F}_{t\wedge\tau-} := \mathcal{F}_t \cap \mathcal{F}_{\tau-}$. In the following we may also write $(\mathcal{F}_t)_{0\le t<\tau}$ for the filtration on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$.

Working with standard systems will allow us to derive for every strictly positive strict local P-martingale the existence of a measure Q on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_t)_{0\le t<\tau})$, such that the reciprocal of the strict local P-martingale is a true Q-martingale. In Section 2.4 we will use this result to reduce calculations involving strict local martingales to the much easier case of true martingales.

From Theorem 4 in [16] and Proposition 6 in [60] we know that every continuous local martingale understood as the canonical process on $C(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ gives rise to a new measure under which its reciprocal turns into a true martingale. In the context of arbitrage theory similar results have recently been derived and applied by [28] and [68] for continuous processes in a Markovian setting. Theorem 2.2.12 below is an extension of these results to more general probability spaces and càdlàg processes. Its proof relies on the construction of the Föllmer measure, cf. [29] and [54]; nevertheless we will give a detailed proof, since it is essential for the rest of this chapter and the next chapter as well.

Proposition 2.2.8. *Let $\left(\Omega, \mathcal{F}, (\widetilde{\mathcal{F}}_t)_{t\ge0}, \mathsf{P}\right)$ be a filtered probability space and assume that $(\widetilde{\mathcal{F}}_t)_{t\ge0}$ is a standard system. Let X be a càdlàg local martingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathsf{P})$ with values in $(0,\infty)$ and $X_0 = 1$ P-almost surely. We define $\tau^X_n := \inf\{t \ge 0 :\ X_t > n\} \wedge n$ and $\tau^X = \lim_{n\to\infty} \tau^X_n$. Then there exists a unique probability measure Q on $\left(\Omega, \mathcal{F}_{\tau^X-}, (\mathcal{F}_{t\wedge\tau^X-})_{t\ge0}\right)$, such that $\left.\frac{d\mathsf{P}}{d\mathsf{Q}}\right|_{\mathcal{F}_t\cap\mathcal{F}_{\tau^X-}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$ for all $t \ge 0$. Moreover, $1/X$ is a local Q-martingale on the interval $[0, \tau^X)$ which does not jump to zero Q-almost surely.*

Proof. First, note that τ^X_n is an $(\mathcal{F}_t)_{t\ge0}$-stopping time and the process $(X_{t\wedge\tau^X_n})_{t\ge0}$ is a uniformly integrable $\{(\mathcal{F}_t)_{t\ge0}, \mathsf{P}\}$-martingale for all $n \in \mathbb{N}$. Indeed, if (σ_m) is any localizing sequence for X such that $\mathbb{E}^{\mathsf{P}} X_{\sigma_m} = 1$ for all $m \in \mathbb{N}$, then

$$X_{\tau^X_n\wedge\sigma_m} \le n \vee X_{\tau^X_n} \qquad \text{and} \qquad \mathbb{E}^{\mathsf{P}}(n \vee X_{\tau^X_n}) \le n + \mathbb{E}^{\mathsf{P}} X_{\tau^X_n} \le n + 1$$

by the supermartingale property of X. By the dominated convergence theorem we thus conclude that $\mathbb{E}^{\mathsf{P}} X_{\tau_n^X} = 1$ and thus (τ_n^X) is a localizing sequence as well.
Furthermore, $\mathsf{P}(\tau^X = \infty) = 1$, since a positive càdlàg local martingale does not explode almost surely. We define on $(\Omega, \mathcal{F}_{\tau_n^X})$ the probability measure $\widetilde{\mathsf{Q}}_n$ via $\widetilde{\mathsf{Q}}_n = X_{\tau_n^X}.\mathsf{P}|_{\mathcal{F}_{\tau_n^X}}$ for all $n \in \mathbb{N}$. The family $(\widetilde{\mathsf{Q}}_n)_{n\in\mathbb{N}}$ constitutes a consistent family of probability measures on $(\mathcal{F}_{\tau_n^X})_{n\geq 1}$: If $A \in \mathcal{F}_{\tau_n^X}$, then

$$\widetilde{\mathsf{Q}}_{n+k}(A) = \mathbb{E}^{\mathsf{P}}\left(X_{\tau_{n+k}^X}\mathbb{1}_A\right) = \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}\mathbb{1}_A) = \widetilde{\mathsf{Q}}_n(A),$$

i.e. $\widetilde{\mathsf{Q}}_{n+k}|_{\mathcal{F}_{\tau_n^X}} = \widetilde{\mathsf{Q}}_n$ for all $n, k \in \mathbb{N}$. This induces a sequence of consistently defined measures $(\mathsf{Q}_n)_{n\in\mathbb{N}}$ on the sequence $(\mathcal{F}_{\tau_n^X-})_{n\in\mathbb{N}}$, which is a standard system by Lemma 2.2.7. Note that $\mathcal{F}_{\tau^X-} = \bigvee_{n\geq 1}\mathcal{F}_{\tau_n^X-}$, since $(\tau_n^X)_{n\geq 1}$ is increasing. We can thus apply Theorem 3.2 together with Theorem 4.1 in Chapter V of [61], cf. also Theorem 6.2 in [29], which yield the existence of a unique measure Q on $\left(\Omega, \mathcal{F}_{\tau^X-}, (\mathcal{F}_{t\wedge\tau^X-})_{t\geq 0}\right)$ such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X-}} = \mathsf{Q}_n = \widetilde{\mathsf{Q}}_n|_{\mathcal{F}_{\tau_n^X-}}$. Moreover, since $\{\tau_n^X < \tau_m^X\} \in \mathcal{F}_{\tau_m^X-}$,

$$\begin{aligned}
\mathsf{Q}(\tau_n^X < \tau^X) &= \lim_{m\to\infty}\mathsf{Q}(\tau_n^X < \tau_m^X) = \lim_{m\to\infty}\widetilde{\mathsf{Q}}_m(\tau_n^X < \tau_m^X) = \lim_{m\to\infty}\mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{\tau_n^X<\tau_m^X\}}X_{\tau_m^X}\right)\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{\tau_n^X<\tau_m^X\}}X_{\tau_n^X}\right) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{\tau_n^X<\tau^X\}}X_{\tau_n^X}\right) = \mathbb{E}^{\mathsf{P}}\left(X_{\tau_n^X}\right) = 1,
\end{aligned}$$

i.e. $1/X$ does not jump to zero under Q. Therefore, if $\Lambda_n \in \mathcal{F}_{\tau_n^X}$, then

$$\begin{aligned}
\mathsf{Q}(\Lambda_n) &= \mathsf{Q}\left(\Lambda_n \cap \{\tau^X > \tau_n^X\}\right) = \lim_{m\to\infty}\mathsf{Q}\left(\Lambda_n \cap \{\tau_m^X > \tau_n^X\}\right)\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathsf{P}}\left(X_{\tau_m^X}\mathbb{1}_{\Lambda_n}\mathbb{1}_{\{\tau_m^X>\tau_n^X\}}\right) = \lim_{m\to\infty}\mathbb{E}^{\mathsf{P}}\left(X_{\tau_n^X}\mathbb{1}_{\Lambda_n}\mathbb{1}_{\{\tau_m^X>\tau_n^X\}}\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(X_{\tau_n^X}\mathbb{1}_{\Lambda_n}\right) = \widetilde{\mathsf{Q}}_n(\Lambda_n).
\end{aligned}$$

Therefore, $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} = \widetilde{\mathsf{Q}}_n$ for all $n \in \mathbb{N}$.
Now let S be an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Note that $\{S < \tau_n^X\} \in \mathcal{F}_S$ and $\{S < \tau_n^X\} \in \mathcal{F}_{\tau_n^X}$. Thus,

$$\begin{aligned}
\mathsf{Q}(S < \tau_n^X) &= \widetilde{\mathsf{Q}}_n(S < \tau_n^X) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{S<\tau_n^X\}}X_{\tau_n^X}\right) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{S<\tau_n^X\}}\mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}|\mathcal{F}_S)\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{S<\tau_n^X\}}X_S\right).
\end{aligned}$$

Since $\mathsf{P}(\tau_n^X < \tau^X = \infty) = 1$, taking the limit as $n \to \infty$ in the above equation yields

$$\mathsf{Q}(S < \tau^X) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{S<\infty\}}X_S\right). \tag{2.1}$$

Applied to the stopping time $S_A := S\mathbb{1}_A + \infty\mathbb{1}_{A^c}$, where $A \in \mathcal{F}_S$, this gives

$$\mathsf{Q}(S < \tau^X, A) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{A\cap\{S<\infty\}}X_S\right).$$

Especially, if S is finite P-almost surely, then $\mathsf{Q}(S < \tau^X, A) = \mathbb{E}^{\mathsf{P}}(X_S\mathbb{1}_A)$ for $A \in \mathcal{F}_S$.
If $A \in \mathcal{F}_t \cap \mathcal{F}_{\tau^X-}$, then

$$\begin{aligned}
\mathsf{P}(A) &= \lim_{n\to\infty}\mathsf{P}(A \cap \{t < \tau_n^X\}) = \lim_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A\mathbb{1}_{\{t<\tau_n^X\}}\frac{1}{X_{\tau_n^X}}\right)\\
&= \lim_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A\mathbb{1}_{\{t<\tau_n^X\}}\frac{1}{X_t}\right) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A\mathbb{1}_{\{t<\tau^X\}}\frac{1}{X_t}\right).
\end{aligned}$$

Therefore, $\frac{d\mathsf{P}}{d\mathsf{Q}}\Big|_{\mathcal{F}_t \cap \mathcal{F}_{\tau^X-}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$ for all $t \geq 0$.

Finally, note that because $(X_t^{\tau_n^X})_{t\geq 0}$ is a strictly positive uniformly integrable P-martingale for all $n \in \mathbb{N}$, $\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} \sim \mathsf{Q}|_{\mathcal{F}_{\tau_n^X}}$ and

$$d\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} = \frac{1}{X_{\tau_n^X}} d\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} \quad \Leftrightarrow \quad \frac{d\mathsf{Q}}{d\mathsf{P}}\Big|_{\mathcal{F}_{t\wedge\tau_n^X}} = X_{t\wedge\tau_n^X} \;\forall\; t \geq 0.$$

Thus,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}}\middle|\mathcal{F}_s\right) = \mathbb{E}^{\mathsf{P}}\left(\frac{1}{X_{t\wedge\tau_n^X}} \cdot \frac{X_{t\wedge\tau_n^X}}{X_{s\wedge\tau_n^X}}\middle|\mathcal{F}_s\right) = \frac{1}{X_{s\wedge\tau_n^X}}$$

for $s \leq t$, i.e. $\frac{1}{X}$ is a local Q-martingale on the interval $\bigcup_{n\in\mathbb{N}}[0,\tau_n^X] = [0,\tau^X)$. □

Corollary 2.2.9. *Under the assumptions of Proposition 2.2.8, X is a strict local P-martingale, if and only if $\mathsf{Q}(\tau^X < \infty) > 0$.*

Proof. It follows directly from equation (2.1) that $\mathsf{Q}(t < \tau^X) = \mathbb{E}^{\mathsf{P}} X_t$, which is smaller than 1 for some t, iff X is a strict local martingale under P. □

Remark 2.2.10. Corollary 2.2.9 makes clear why we cannot work with the natural augmentation of $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$. Indeed, we have $A_n := \{\tau^X \leq n\} \in \mathcal{F}_n \cap \mathcal{F}_{\tau^X-}$ and $\mathsf{P}(A_n) = 0$ for all $n \in \mathbb{N}$, while $\mathsf{Q}(A_n) > 0$ for some n if X is a strict local P-martingale. However, it is in general rather inconvenient to work without any augmentation, especially if one works with an uncountable number of stochastic processes. For this reason a new kind of augmentation - called the (τ_n^X)-natural augmentation - is introduced in Chapter 3, which is suitable for the change of measure from P to Q undertaken here. Since for the financial applications in the second part of this chapter the setup introduced above is already sufficient, we do not bother about this augmentation here and refer the interested reader to Chapter 3 for more technical details.

In the following we extend the measure Q in an arbitrary way from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}_\infty = \bigvee_{t\geq 0}\widetilde{\mathcal{F}}_t$. For notational convenience we assume that $\mathcal{F} = \mathcal{F}_\infty$. In fact it is always possible to extend a probability measure from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$: since $(\Omega, \widetilde{\mathcal{F}}_t)$ is a standard Borel space for every $t \geq 0$ and $(\Omega, \mathcal{F}_{\tau_n^X-})$ is a standard Borel space for all $n \in \mathbb{N}$ by Lemma 2.2.7, it follows from Theorem 4.1 in [61] that $(\Omega, \mathcal{F})$ and $(\Omega, \mathcal{F}_{\tau^X-})$ are also standard Borel spaces. Especially, they are countably generated which allows us to apply Theorem 3.1 of [26] that guarantees an extension of Q from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$. Moreover, it does not matter for the results how we extend it, because all events that happen with positive probability under P take place before time τ^X under Q almost surely. However, if Y is any process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, then Y_t is only defined on $\{t < \tau^X\}$ under Q. Especially, if Y is a P-semimartingale, then $Y^{\tau_n^X}$ is a Q-semimartingale for each $n \in \mathbb{N}$ as follows from Girsanov's theorem, since $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} \sim \mathsf{P}|_{\mathcal{F}_{\tau_n^X}}$. Therefore, Y is a Q-semimartingale on the stochastic interval $\bigcup_{n\in\mathbb{N}}[0,\tau_n^X]$ or a "semimartingale up to time τ^X" in the terminology of [35]. We note that in general it may not be possible to extend Y to the whole positive real line under Q in such a way that Y remains a semimartingale. Indeed, according to Proposition 5.8 of [35] such an extension is possible if and only if Y_{τ^X-} exists in $\mathbb{R}_+$ Q-almost surely. We define the process $\widetilde{Y}$ as

$$\widetilde{Y}_t = \begin{cases} Y_t & : \; t < \tau^X \\ \liminf_{s\to\tau^X, s<\tau^X, s\in\mathbb{Q}} Y_s & : \; \tau^X \leq t < \infty \end{cases}. \tag{2.2}$$

Note that $\widetilde{Y}_t = Y_t$ on $\{t < \tau^X\}$. The above definition specifies an extension of the process Y, which is a priori only defined up to time τ^X, to the whole positive real line. In the following we will work with this extension.

Lemma 2.2.11. *Under the assumptions of Proposition 2.2.8 we have* $\frac{1}{\widetilde{X}_t} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$. *Furthermore, the process* $\left(\frac{1}{\widetilde{X}_t}\right)_{t\geq 0}$ *is a true* Q*-martingale for any extension of* Q *from* $\mathcal{F}_{\tau^X-}$ *to* $\mathcal{F}$.

Proof. First note that Q-almost surely

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}} &= \limsup_{n\to\infty}\left(\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}} + \frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{t\geq\tau_n^X\}}\right)\\ &\leq \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}} + \limsup_{n\to\infty}\frac{1}{n}\mathbb{1}_{\{t\geq\tau_n^X\}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}\end{aligned}$$

and

$$\liminf_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}} = \liminf_{n\to\infty}\left(\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}} + \frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{t\geq\tau_n^X\}}\right) \geq \liminf_{n\to\infty}\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$$

Thus, $\frac{1}{\widetilde{X}_t} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$. Furthermore,

$$\begin{aligned}0 &\leq \frac{1}{X_{\tau^X-}}\mathbb{1}_{\{\tau^X<\infty\}} = \lim_{k\to\infty}\frac{1}{X_{\tau^X-}}\mathbb{1}_{\{\tau^X<k\}} = \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{\tau^X<k\}}\\ &\leq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathbb{1}_{\{\tau^X<k\}} = 0\end{aligned}$$

implies that $X_{\tau^X-} = \infty$ on $\{\tau^X < \infty\}$ Q-almost surely. From the proof of Proposition 2.2.8 we know that $\frac{1}{X^{\tau_n^X}}$ is a true Q-martingale for all $n \in \mathbb{N}$. By the definition of τ_n^X we have for any integer $n \geq t$:

$$X_{t\wedge\tau_n^X} = \widetilde{X}_{t\wedge\tau_n^X} = \widetilde{X}_{t\wedge\inf\{s\geq 0:\ \widetilde{X}_s>n\}} \geq \widetilde{X}_t\wedge 1 \qquad \Rightarrow \qquad \frac{1}{X_{t\wedge\tau_n^X}} \leq \frac{1}{\widetilde{X}_t\wedge 1} = 1\vee\frac{1}{\widetilde{X}_t}.$$

Because

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\widetilde{X}_t}\right) = \mathbb{E}^{\mathsf{Q}}\left(\liminf_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}}\right) \leq \liminf_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}}\right) = 1,$$

the dominated convergence theorem implies that for all $0 \leq s \leq t$

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\widetilde{X}_t}\middle|\mathcal{F}_s\right) = \mathbb{E}^{\mathsf{Q}}\left(\lim_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}}\middle|\mathcal{F}_s\right) = \lim_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}}\middle|\mathcal{F}_s\right) = \lim_{n\to\infty}\frac{1}{X_{s\wedge\tau_n^X}} = \frac{1}{\widetilde{X}_s}.$$

□

To simplify notation we identity in the following the process X with $\widetilde{X}$. We summarize our results so far in the following theorem:

Theorem 2.2.12. *Let* $\left(\Omega,\mathcal{F},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\mathsf{P}\right)$ *be a filtered probability space and assume that* $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ *is a standard system. Let* X *be a càdlàg local martingale on* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathsf{P})$ *with values in* $(0,\infty)$ *and* $X_0 = 1$ P*-almost surely. We define* $\tau_n^X := \inf\{t\geq 0:\ X_t > n\}\wedge n$ *and* $\tau^X = \lim_{n\to\infty}\tau_n^X$. *Then there exists a probability measure* Q *on* $(\Omega,\mathcal{F}_\infty)$ *such that* $1/X$ *is a* Q*-martingale, which does not jump to zero* Q*-almost surely, and such that* $\mathsf{Q}(A,\tau^X > t) = \mathbb{E}^{\mathsf{P}}(X_t\mathbb{1}_A)$ *for all* $t \geq 0$ *and* $A \in \mathcal{F}_t$. *In particular,* $\mathsf{P}|_{\mathcal{F}_t} \ll \mathsf{Q}|_{\mathcal{F}_t}$ *for all* $t \geq 0$.

Note that in the case where X is a strict local P-martingale Theorem 2.2.12 is a precise converse to Theorem 1.1, if one identifies X of Theorem 2.2.12 with $1/Y$ of Theorem 2.2.1.

2.3 Examples

In this section we shed new light on some known examples of strict local martingales by applying the theory from the last section for illustration.

2.3.1 Continuous local martingales

For the following examples we work on the path space $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ with W denoting the coordinate process. Here, $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous augmentation of the canonical filtration generated by the coordinate process and P is Wiener measure.

Exponential local martingales

Suppose that X has dynamics

$$dX_t = X_t b(Y_t) dW_t, \quad X_0 = 1,$$

where Y is assumed to be a (possibly explosive) diffusion with

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = y \in \mathbb{R}.$$

Here $b(\cdot), \mu(\cdot)$ and $\sigma(\cdot)$ are chosen such that both SDEs allow for strong solutions and guarantee X to be strictly positive. Exponential local martingales of this type are further studied in [55]. Under Q the dynamics of $\frac{1}{X}$ up to time τ^X are

$$d\left(\frac{1}{X_t}\right) = -\frac{b(Y_t)}{X_t} dW_t^{\mathsf{Q}}$$

for a Q-Brownian motion W^{Q} defined up to time τ^X, and the Q-dynamics of Y_t up to time τ^X are

$$dY_t = \left[\mu(Y_t) + \sigma(Y_t)b(Y_t)\right] dt + \sigma(Y_t) dW_t^{\mathsf{Q}}.$$

Notably, the criterion whether X is a strict local or a true P-martingale from [55], Theorem 2.1, is deterministic and only involves the functions b, σ and μ via the scale function of the original diffusion Y under P and an auxiliary diffusion $\widetilde{Y}$, whose dynamics are identical with the Q-dynamics of Y stated above.

Diffusions in natural scale

We now take X to be a local P-martingale of the form

$$dX_t = \sigma(X_t)dW_t, \quad X_0 = 1,$$

assuming that $\sigma(x)$ is locally bounded and bounded away from zero for $x > 0$ and $\sigma(0) = 0$. Using the results from [20], we know that X is strictly positive, whenever

$$\int_0^1 \frac{x}{\sigma^2(x)} dx = \infty,$$

which we shall assume in the following. Furthermore, X is a strict local martingale, if and only if

$$\int_1^\infty \frac{x}{\sigma^2(x)} dx < \infty.$$

We know that $\frac{1}{X}$ is a Q-martingale, where $\left.\frac{d\mathsf{P}}{d\mathsf{Q}}\right|_{\mathcal{F}_t} = \frac{1}{X_t}$, with decomposition

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2} dW_t^{\mathsf{Q}} = \overline{\sigma}\left(\frac{1}{X_t}\right) dW_t^{\mathsf{Q}}$$

for a Q-Brownian motion W^{Q} defined up to time τ^X and $\overline{\sigma}(y) := -y^2 \cdot \sigma\left(\frac{1}{y}\right)$. Note that

$$\int_1^\infty \frac{y}{\overline{\sigma}^2(y)} dy = \int_0^1 \frac{x}{\sigma^2(x)} dx = \infty,$$

which confirms that $\frac{1}{X}$ is a true Q-martingale. We see that, if X is a strict local martingale under P, then

$$\int_0^1 \frac{y}{\overline{\sigma}^2(y)} dy = \int_1^\infty \frac{x}{\sigma^2(x)} dx < \infty,$$

i.e. $\frac{1}{X}$ hits zero in finite time Q-almost surely.

2.3.2 Jump example

[3] Let $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}})$ with $(\xi_t)_{t\geq 0}$ denoting the coordinate process and $(\mathcal{F}_t)_{t\geq 0}$ being the right-continuous augmentation of the canonical filtration generated by the coordinate process. Assume that under P, $(\xi_t)_{t\geq 0}$ is a one-dimensional Lévy process with $\xi_0 = 0$, $\mathbb{E}^{\mathsf{P}} \exp(b\xi_t) = \exp(t\rho(b)) < \infty$ for all $t \geq 0$ and characteristic exponent

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}}\right) \pi(dx),$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and π is a positive measure on $\mathbb{R}\backslash\{0\}$ such that $\int (1 \wedge |x|^2)\pi(dx) < \infty$. Define

$$X_t = Y_t^b \exp\left(-\rho(b) \int_0^t \frac{ds}{Y_s}\right),$$

where $(Y_t)_{t\geq 0}$ is a semistable Markov process, i.e. $\left(\frac{1}{c} Y_{ct}^{(x)}\right)_{t\geq 0} \overset{(d)}{=} \left(Y_t^{(xc^{-1})}\right)_{t\geq 0}$ for all $c > 0$, implicitly defined via

$$\exp(\xi_t) = Y_{\int_0^t \exp(\xi_s) ds}.$$

Following [12], $(X_t)_{t\geq 0}$ is a positive strict local martingale if a and b satisfy

$$-a + \int_{|x|>1} x\pi(dx) \geq 0, \qquad -a + b\sigma^2 - \int_{|x|<1} x(1 - e^{bx})\pi(dx) + \int_{|x|>1} xe^{bx}\pi(dx) < 0.$$

Furthermore, under the new measure Q the process

$$\frac{1}{X_t} = Y_t^{-b} \exp\left(\rho(b) \int_0^t \frac{ds}{Y_s}\right)$$

is a true martingale, where now $(\xi_t)_{t\geq 0}$ has characteristic exponent $\widetilde{\Psi}$ with

$$\widetilde{\Psi}(u) = \Psi(u - ib) - \Psi(-ib).$$

[3] This example is taken from [12]. However, we corrected a small mistake concerning the time-scaling.

2.4 Application to financial bubbles I: decomposition formulas

In this section we apply our results to option pricing in the presence of strict local martingales. For this, we assume that the following standing assumption **(S)** holds throughout the entire section:

(S) *X is assumed to be a càdlàg strictly positive local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, whose filtration is the right-continuous augmentation of a standard system and $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We assume that $X_0 = 1$ and set $\tau_n^X = \inf\{t \geq 0 |\ X_t > n\} \wedge n$ for all $n \in \mathbb{N}$ and $\tau^X = \lim_{n\to\infty} \tau_n^X$. Furthermore, we denote by Q any extension to $(\Omega, \mathcal{F})$ of the measure associated with X, defined in Theorem 2.2.12.*

We consider a financial market model which satisfies the NFLVR property as defined in [17]. We denote by P an equivalent local martingale measure (ELMM). Assuming that the interest rate equals zero, we interpret X as the (discounted) stock price process, which is a local martingale under P. In this context, the question of whether X is a strict local or a true P-martingale determines whether there exists a stock price bubble. If X is a strict local P-martingale, the fundamental value of the asset (given by the conditional expectation) deviates from its actual market price X. Several authors, cf. e.g. [15, 38, 39, 60], have interpreted this as the existence of a stock price bubble, which we formally define as follows:

Definition 2.4.1. With the previous notation the *asset price bubble* for the stock price process X between time $t \geq 0$ and time $T \geq t$ is equal to the $\mathcal{F}_t$-measurable random variable
$$\gamma_X(t,T) := X_t - \mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_t).$$

Remark 2.4.2. For $t = 0$ we recover the 'default' function $\gamma_X(0,T) = X_0 - \mathbb{E}^{\mathsf{P}} X_T$ of the local martingale X, which was introduced in [25]. Here the term 'default' refers to the locality property of X and measures its failure of being a martingale. In [24, 25] the authors derive several expressions for the default function in terms of the first hitting time, the local time and the last passage time of the local martingale.

Remark 2.4.3. Note that the above definition of a bubble depends on the measure P, which may be viewed as the subjective valuation measure of a certain economic agent. From the agent's point of view, the asset price contains a bubble. Only in a complete market, i.e. if and only if P is the unique ELMM, the notion of a bubble becomes universal without any element of subjectivity.

In Proposition 7 of [60] the price of a non-path-dependent option written on a stock, whose price process is a (strict) local martingale, is decomposed into a "normal" ("non-bubble") term and a default term. In the following we give an extension of this theorem to a certain class of path-dependent options. For this let us introduce the following notation for all $k \in \mathbb{N}$:
$$\mathbb{R}_+^k = \{x \in \mathbb{R}^k :\ x_l \geq 0,\ l = 1, \ldots, k\}, \quad \mathbb{R}_{++}^k = \{x \in \mathbb{R}^k :\ x_l > 0,\ l = 1, \ldots, k\}.$$

Theorem 2.4.4. *Let $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and consider a Borel-measurable non-negative function $h : \mathbb{R}_{++}^n \to \mathbb{R}_+$. Define the function $g(x) := x_n \cdot h\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}_{++}^n$. Then:*
$$\mathbb{E}^{\mathsf{P}} h\left(X_{t_1}, \ldots, X_{t_n}\right) = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_{t_1}}, \ldots, \frac{1}{X_{t_n}}\right) \mathbb{1}_{\{\tau^X > t_n\}}\right).$$

Now suppose that the following limits exist in $\mathbb{R}_+$ *for* $y_i \in \mathbb{R}_{++},\ i = 1, \dots, n-1$:

$$\begin{aligned}\lim_{|z|\to 0} g\left(y_1, \dots, y_k; z_1, \dots, z_{n-k}\right) &=: \eta_k(y_1, \dots, y_k), \quad k = 1, \dots, n-1,\\ \lim_{|z|\to 0} g\left(z_1, \dots, z_n\right) &=: \eta_0.\end{aligned}$$

Define $\overline{g} : A \to \mathbb{R}_+$ *as the extension of* g *from* $\mathbb{R}^n_{++}$ *to* $A \subset \mathbb{R}^n_+$, *where* A *is defined as* $A := \{x \in \mathbb{R}^n_+ :$ *if* $x_k = 0$ *for some* $k = 1, \dots, n$, *then* $x_l = 0\ \forall\ l \geq k\}$. *Then:*

$$\mathbb{E}^{\mathsf{P}} h\left(X_{t_1}, \dots, X_{t_n}\right) = \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_n}}\right) - \sum_{k=0}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \leq t_{k+1}\}} \cdot \eta_k(X^k)\right), \tag{2.3}$$

where we set $t_0 = 0$ *and* $X^k = \left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_k}}\right)$ *for* $k = 1, \dots, n-1$, $X^0 \equiv 0$.
In particular, if $\eta_k(\cdot) \equiv c_k,\ k = 1, \dots, n-1$, *are constant, then:*

$$\mathbb{E}^{\mathsf{P}} h\left(X_{t_1}, \dots, X_{t_n}\right) = \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_n}}\right) - \sum_{k=0}^{n-1} c_k \cdot \mathsf{Q}\left(t_k < \tau^X \leq t_{k+1}\right). \tag{2.4}$$

Proof. First note that

$$\mathbb{1}_{\{\tau^X > t_n\}} = \mathbb{1}_{\{\tau^X > t_1\}} \mathbb{1}_{\{\tau^X > t_2\}} \dots \mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{1}_{\{\tau^X > t_n\}}.$$

Using the change of measure $d\mathsf{P}|_{\mathcal{F}_{t_n}} = \frac{1}{X_{t_n}} d\mathsf{Q}|_{\mathcal{F}_{t_n}}$ on $\{\tau^X > t_n\}$ we deduce

$$\begin{aligned}&\mathbb{E}^{\mathsf{P}} h(X) = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X}\right)\mathbb{1}_{\{\tau^X > t_n\}}\right) = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X}\right)\mathbb{1}_{\{\tau^X > t_1\}} \dots \mathbb{1}_{\{\tau^X > t_n\}}\right) = \\ &\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_2\}} \dots \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_n\}} g\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-1}}\right)\middle| \mathcal{F}_{t_{n-2}}\right) \dots \middle| \mathcal{F}_{t_2}\right)\middle| \mathcal{F}_{t_1}\right)\right).\end{aligned}$$

Because on $\{\tau^X > t_{n-1}\}$ we have

$$\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_n\}} g\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-1}}\right) = \mathbb{E}^{\mathsf{Q}}\left(\overline{g}\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-1}}\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-1} < \tau^X \leq t_n\}} \eta_{n-1}\left(X^{n-1}\right)\middle| \mathcal{F}_{t_{n-1}}\right),$$

it follows that

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) = \mathbb{E}^{\mathsf{Q}}&\left(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_2\}} \dots \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-2}\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \overline{g}\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-2}}\right) \dots \middle| \mathcal{F}_{t_1}\right)\right)\right)\\ &- \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{1}_{\{\tau^X > t_2\}} \dots \mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{1}_{\{t_{n-1} < \tau^X \leq t_n\}} \eta_{n-1}\left(X^{n-1}\right)\right).\end{aligned}$$

Similarly, on $\{\tau^X > t_{n-2}\}$ we have

$$\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \overline{g}\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-2}}\right) = \mathbb{E}^{\mathsf{Q}}\left(\overline{g}\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-2}}\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-2} < \tau^X \leq t_{n-1}\}} \eta_{n-2}\left(X^{n-2}\right)\middle| \mathcal{F}_{t_{n-2}}\right),$$

and we deduce that

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) \;=\; &\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_2\}} \dots \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-2}\}} \overline{g}\left(\frac{1}{X}\right)\middle| \mathcal{F}_{t_{n-3}}\right) \dots \middle| \mathcal{F}_{t_1}\right)\right)\\ &- \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-2} < \tau^X \leq t_{n-1}\}} \eta_{n-2}\left(X^{n-2}\right)\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-1} < \tau^X \leq t_n\}} \eta_{n-1}\left(X^{n-1}\right)\right).\end{aligned}$$

Iterating this procedure results in

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) &= \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \overline{g}\left(\frac{1}{X}\right)\right) - \sum_{k=1}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \leq t_{k+1}\}} \eta_k \left(X^k\right)\right) \\ &= \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X}\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \leq t_1\}} \eta_0\right) - \sum_{k=1}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \leq t_{k+1}\}} \eta_k \left(X^k\right)\right).\end{aligned}$$

□

Remark 2.4.5. The sum following the minus sign in the above decompositions (2.3) and (2.4) will be called the *default term*. This is motivated by the following observation:

$$\gamma_X(t,T) = X_t - \mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_t) = X_t - X_t \cdot \mathsf{Q}(\tau^X > T|\mathcal{F}_t) = X_t \cdot \mathsf{Q}(\tau^X \leq T|\mathcal{F}_t) \quad \mathsf{P}\text{-a.s.} \tag{2.5}$$

Here, the second equality in (2.5) is justified by the following calculation, valid for any $\mathcal{F}_t$-measurable set A:

$$\begin{aligned}\mathbb{E}^{\mathsf{P}}(\mathbb{1}_A X_T) &= \mathsf{Q}(A, \tau^X > T) = \mathsf{Q}(A, \tau^X > t, \tau^X > T) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{A, \tau^X > t\}} \mathsf{Q}(\tau^X > T|\mathcal{F}_t)\right) \\ &= \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_A X_t \cdot \mathsf{Q}(\tau^X > T|\mathcal{F}_t)\right) \quad \mathsf{P}\text{-a.s.}\end{aligned}$$

Taking expectations with respect to P in (2.5) yields

$$\mathbb{E}^{\mathsf{P}} \gamma_X(t,T) = \mathbb{E}^{\mathsf{P}}\left(X_t \cdot \mathsf{Q}(\tau^X \leq T|\mathcal{F}_t)\right) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t\}} \mathsf{Q}(\tau^X \leq T|\mathcal{F}_t)\right) = \mathsf{Q}(t < \tau^X \leq T).$$

Thus, the default term is directly related to the expected bubble of the underlying. It measures how much the failure of the martingale property by X affects the option price. If X is a true martingale it will equal zero.

The convergence conditions that must be fulfilled in Theorem 2.4.4 may seem to be rather strict. However, below we give a few examples of options which satisfy those conditions.

Example 2.4.6. Let us consider a modified call option with maturity T and strike K, where the holder has the option to reset the strike value to the current stock price at certain points in time $t_1 < t_2 < \cdots < t_n < T$, i.e. the payoff profile of the option is given by

$$H(X) = (X_T - \min(K, X_{t_1}, X_{t_2}, \ldots, X_{t_n}))^+.$$

With the notation in Theorem 2.4.4 and setting $t_{n+1} = T$ it follows that

$$\eta_0 = \eta_1 = \cdots = \eta_n = 1$$

and the option value can be decomposed as

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) &= \mathbb{E}^{\mathsf{Q}}\left(1 - \frac{1}{X_T} \cdot \min\left(K, X_{t_1}, \ldots, X_{t_n}\right)\right)^+ - \sum_{k=0}^{n} \mathsf{Q}\left(t_k < \tau^X \leq t_{k+1}\right) \\ &= \mathbb{E}^{\mathsf{Q}}\left(1 - \frac{1}{X_T} \cdot \min\left(K, X_{t_1}, \ldots, X_{t_n}\right)\right)^+ - \gamma_X(0,T).\end{aligned}$$

Therefore, this modified call option has the same default as the normal call option, cf. equation (14) in [60].

Example 2.4.7. Let us consider a call option on the ratio of the stock price at times T and $S \leq T$ with strike $K \in \mathbb{R}_+$, i.e.

$$h(X) = \left(\frac{X_T}{X_S} - K \right)^+$$

for $S < T \in \mathbb{R}_+$. In this case

$$\eta_0 = 0, \qquad \eta_1(y) = y$$

and the decomposition of the option value is given by

$$\mathbb{E}^{\mathsf{P}} h(X) = \mathbb{E}^{\mathsf{Q}} \left(\frac{1}{X_S} - \frac{K}{X_T} \right)^+ - \mathbb{E}^{\mathsf{Q}} \left(\mathbb{1}_{\{S < \tau^X \leq T\}} \frac{1}{X_S} \right).$$

Example 2.4.8. A *chooser option* with maturity T and strike K entitles the holder to decide at time $S < T$, whether the option is a call or a put. He will choose the call, if its value is as least as high as the value of the put option with strike K and maturity T at time S. However, in the presence of asset price bubbles, i.e. when the underlying is a strict local martingale, put-call-parity does not hold, but instead we have

$$\mathbb{E}^{\mathsf{P}}((X_T - K)^+ | \mathcal{F}_S) - \mathbb{E}^{\mathsf{P}}((K - X_T)^+ | \mathcal{F}_S) = \mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_S) - K.$$

Therefore, the payoff of the chooser option equals

$$h(X_S, X_T) = (X_T - K)^+ \mathbb{1}_{\{\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S) \geq K\}} + (K - X_T)^+ \mathbb{1}_{\{\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S) < K\}}.$$

Let us assume that X is Markovian. Then we can express $\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S)$ as a function of X_S, say $\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S) = m(X_S)$, and the limits defined in Theorem 2.4.4 exist, if m is monotone for large values, and equal

$$\eta_1(y) = \mathbb{1}_{\left\{m\left(\frac{1}{y}\right) \geq K\right\}}, \qquad \eta_0 = \lim_{x \to \infty} \mathbb{1}_{\{m(x) \geq K\}}.$$

Thus, the value of the chooser option can be decomposed as

$$\begin{aligned} \mathbb{E}^{\mathsf{P}} h(X_S, X_T) &= \mathbb{E}^{\mathsf{Q}} \left(\frac{h(X_S, X_T)}{X_T} \right) - \mathsf{Q}\left(m(X_S) \geq K,\ S < \tau^X \leq T \right) \\ &\quad - \lim_{x \to \infty} \mathbb{1}_{\{m(x) \geq K\}} \mathsf{Q}(\tau^X \leq S) \ . \end{aligned}$$

If X is the reciprocal of a BES(3)-process under P, it is calculated in subsection 2.2.2 in [15] that

$$m(X_S) = \mathbb{E}^{\mathsf{P}}(X_T | X_S) = X_S \left(1 - 2\Phi \left(- \frac{1}{X_S \sqrt{T - S}} \right) \right).$$

Therefore,

$$\lim_{x \to \infty} m(x) = \lim_{x \to \infty} \mathbb{E}^{\mathsf{P}}(X_T | X_S = x) = \lim_{x \to \infty} 2\varphi \left(- \frac{1}{x\sqrt{T - S}} \right) \frac{1}{\sqrt{T - S}} = \frac{\sqrt{2}}{\sqrt{\pi(T - S)}}$$

and

$$\eta_1(y) = \mathbb{1}_{\left\{\frac{1}{y}\left(1 - 2\Phi\left(-\frac{y}{\sqrt{T-S}}\right)\right) \geq K\right\}}, \qquad \eta_0 = \mathbb{1}_{\left\{\frac{\sqrt{2}}{\sqrt{\pi(T-S)}} > K\right\}}.$$

Remark 2.4.9. Here we take the approach of valuating options by risk-neutral expectations. While there may be other approaches, risk-neutral expectations do not create arbitrage in the market, even though the stock itself is not priced that way. Indeed, P remains an ELMM in the enlarged market also after adding any asset $V_t = \mathbb{E}^{\mathsf{P}}[H|\mathcal{F}_t],\ t \leq T$, for some integrable $H \in \mathcal{F}_T$. Interestingly, by choosing $H = X_T$ we may have $V_0 < X_0$ (in the case where X is a strict local martingale). But it is impossible to short X and take a long position on V all the way up to T because of credit constraints, therefore NFLVR is not violated.

In the following we give another extension of Proposition 7 in [60] to Barrier options, i.e. we allow the options to be knocked-in or knocked-out by passing some pre-specified level.

Theorem 2.4.10. *Consider any non-negative Borel-measurable function $h : \mathbb{R}_{++} \to \mathbb{R}_+$ and define $g(x) = x \cdot h\left(\frac{1}{x}\right)$ for $x > 0$. Suppose that $\lim_{x\to 0} g(x) =: \eta < \infty$ exists and denote by $\overline{g} : \mathbb{R}_+ \to \mathbb{R}_+$ the extension of g with $\overline{g}(0) = \eta$. Define $\hat{m}_T^X = \min_{t\leq T} X_t$, $m_T^X = \max_{t\leq T} X_t$ and $\tau_a^X := \inf\{t \geq 0 :\ X_t > a\}$, $T_a^X := \inf\{t \geq 0 :\ X_t \leq a\}$ for $a \in \mathbb{R}_+$. Then for any bounded stopping time T and for any real numbers $D \leq 1$ and $F \geq 1$:*

$$
\begin{aligned}
(DI) \qquad \mathbb{E}^{\mathsf{P}}\left(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\right) &= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\right) - \eta \cdot \mathsf{Q}\left(T_D^X < \tau^X \leq T\right) \\
(DO) \qquad \mathbb{E}^{\mathsf{P}}\left(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \geq D\}}\right) &= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\hat{m}_T^X \geq D\}}\right) - \eta \cdot \mathsf{Q}\left(T_D^X = \infty, \tau^X \leq T\right) \\
(UI) \qquad \mathbb{E}^{\mathsf{P}}\left(h(X_T)\mathbb{1}_{\{m_T^X \geq F\}}\right) &= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{m_T^X \geq F\}}\right) - \eta \cdot \mathsf{Q}\left(\tau^X \leq T\right) \\
(UO) \qquad \mathbb{E}^{\mathsf{P}}\left(h(X_T)\mathbb{1}_{\{m_T^X \leq F\}}\right) &= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{m_T^X \leq F\}}\right)
\end{aligned}
$$

Before proving the theorem we remark that the result is intuitively reasonable because the default only plays a role if the option is active. Especially note that the default term for Up-and-Out options (UO) is equal to zero, since in this case we can replace X by the uniformly integrable martingale $(X_{t\wedge\tau_F^X})$ in the definition of the option's payoff function.

Proof. Keeping in mind that $D \leq 1$ and $F \geq 1$, it follows from the absolute continuity relationship between P and Q that

$$
\begin{aligned}
\mathbb{E}^{\mathsf{P}}\left(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\right) &= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\tau^X > T,\ \hat{m}_T^X \leq D\}}\right) = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\tau^X > T \geq T_D^X\}}\right) \\
&= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\right) - \eta \cdot \mathsf{Q}\left(T_D^X \leq T,\ \tau^X \leq T\right) \\
&= \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_T}\right)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\right) - \eta \cdot \mathsf{Q}\left(T_D^X < \tau^X \leq T\right).
\end{aligned}
$$

This proves the formula for the Down-and-In barrier option (DI). The other three formulas can be proven in a similar way by noting that

$$
\begin{aligned}
\mathsf{Q}\left(\tau^X \leq T < T_D^X\right) &= \mathsf{Q}\left(\tau^X \leq T,\ T_D^X = \infty\right), \\
\mathsf{Q}\left(\tau_F^X \leq T,\ \tau^X \leq T\right) &= \mathsf{Q}\left(\tau^X \leq T\right), \\
\mathsf{Q}\left(\tau^X \leq T < \tau_F^X\right) &= 0.
\end{aligned}
$$

$\square$

Remark 2.4.11. Above we used the risk-neutral pricing approach to calculate the value of some options written on a stock which may have an asset price bubble, as suggested by the first fundamental theorem of asset pricing. The derived decompositions show that there is an important difference in the option value depending on whether the underlying is a strict local or a true martingale under the risk-neutral measure, which is reflected in the default term. Even though we do not create arbitrage opportunities when pricing opions by their fundamental values calculated above, several authors have suggested to "correct" the option price to account for the strictness of the local martingale, cf. e.g. [10, 37, 38, 39, 53]. In [10] the price of a contingent claim is defined as the minimal superreplicating cost under both measures P and Q corresponding to two different currencies, where the process X is interpreted as the exchange rate between them. While the authors of [37, 38, 39] work under the additional No Dominance assumption, which is strictly stronger than NFLVR, and allow for bubbles in the option prices within this framework, in [53] the following pricing formulas for European and American call options written on (continuous) X with strike K and maturity T are suggested:

$$\begin{aligned} C_E^{strict}(K,T) &:= \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{T\wedge\sigma_n} - K)^+, \\ C_A^{strict}(K,T) &:= \sup_{\sigma\in\mathcal{T}_{0,T}} \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\sigma\wedge\sigma_n} - K)^+ \end{aligned}$$

for some localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of the (strict) local martingale X. It is proven in [53] that these definitions are independent of the chosen localizing sequence and that $C_E^{strict} = C_A^{strict}$. However, a generalization of this definition to any other option $h(\cdot)$ on X with maturity T is problematic: the independence of the chosen localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ is not true in general, so one may have to choose $\sigma_n = \tau_n^X$ as defined above. Moreover, in general $\lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} h(X_T^{\sigma_n})$ may not be well-defined and equal to $\mathbb{E}^{\mathsf{P}} h(X_T)$, even when X is a true martingale, as the following example shows.

Example 2.4.12. Suppose that $(\log(X_t)+t/2)_{t\geq 0}$ is a Brownian motion, i.e. X is a geometric Brownian motion, and consider the claim $h(X_T)$ with continuous payoff function

$$h(x) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{n-a_n\leq x\leq n+a_n\}} f_n\left(n - \frac{n|x-n|}{a_n}\right) \quad \text{with} \quad f_n(z) = \frac{1}{\mathsf{P}(\tau_n^X \leq 1)} \cdot \frac{z}{n},$$

where each $a_n \in (0,1)$ is chosen small enough such that

$$2n^2 \cdot \mathsf{P}(n - a_n \leq X_1 \leq n + a_n) \leq \mathsf{P}(\tau_n^X \leq 1).$$

Let us set $T = 1$ and $\sigma_n = \tau_n^X$ for all $n \in \mathbb{N}$. In this case,

$$\mathbb{E}^{\mathsf{P}} h(X_{1\wedge\tau_n^X}) \geq \mathsf{P}(\tau_n^X \leq 1) f_n(n) = 1, \quad n \in \mathbb{N},$$

but

$$\begin{aligned} \mathbb{E}^{\mathsf{P}} h(X_1) &\leq \sum_{n\in\mathbb{N}} \mathsf{P}\left(n - a_n \leq X_1 \leq n + a_n\right) f_n(n) \\ &\leq \sum_{n\in\mathbb{N}} \frac{\mathsf{P}(\tau_n^X \leq 1)}{2n^2} \cdot f_n(n) = \frac{\pi^2}{12} < 1. \end{aligned}$$

Since in this example there are no asset price bubbles, it does not seem correct to trade the option for a price which differs from its fundamental value. Therefore, in the case where we have a decomposition of the fundamental option value as above or more generally as proven in Theorem 2.4.4, this suggests that the most sensible approach to correct the option value for bubbles in the underlying is to set the default term equal to zero. Equivalently, we can also set τ^X equal to infinity under the measure Q. This even gives a way of correcting the option value for stock price bubbles in the general case, where a decomposition formula may not be available, leaving open the question of why this should give an arbitrage-free pricing rule. By doing so we would basically treat the price process as if it were a true martingale. However, we want to emphasize that it is not necessary to correct the price at all, since the fundamental value gives an arbitrage-free price as explained in Remark 2.4.9.

2.5 Relationship between P and Q

In the following we study the relationship between the original measure P and the measure Q in more detail. We suppose that assumption **(S)** is valid throughout the entire section.

Lemma 2.5.1. *Set $X = \widetilde{X}$, i.e. $X_t = \infty$ on $\{t \geq \tau^X\}$. Then, $\mathsf{Q}(X_\infty = \infty) = 1 \Leftrightarrow \mathsf{P}(X_\infty = 0) = 1$.*

Proof. Since X is a P-supermartingale and $\frac{1}{X}$ a Q-martingale, both converge and therefore X_∞ is almost surely well-defined under both measures.
$\underline{\Leftarrow}$: Assume that $\mathsf{P}(X_\infty = 0) = 1$. Because $1/X$ is a Q-martingale, we have by Fatou's lemma for all $u > 0$,

$$\begin{aligned}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X>t,X_t>u\}}\right) &\leq \liminf_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t+n}}\mathbb{1}_{\{\tau^X>t,X_t>u\}}\right) = \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_t}\mathbb{1}_{\{\tau^X>t,X_t>u\}}\right)\\ &= \mathsf{P}(X_t > u).\end{aligned}$$

By dominated convergence for $t \to \infty$,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty,X_\infty>u\}}\right) \leq \mathsf{P}(X_\infty \geq u) = 0 \quad \forall u > 0.$$

This implies that

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty,X_\infty>0\}}\right) = 0.$$

Since $\frac{1}{X}$ is a Q-martingale,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\right) \leq \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_t}\right) = 1.$$

Thus, $\mathsf{Q}(X_\infty = 0) = 0$ and

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty\}}\right) = 0 \;\Leftrightarrow\; \frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty\}} = 0 \;\; \mathsf{Q}\text{-a.s.}$$

Since $\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X<\infty\}} = 0$, it follows that $\frac{1}{X_\infty} = 0$ Q-almost surely.

$\underline{\Rightarrow}$: Assume that $\mathsf{Q}(X_\infty = \infty) = 1$. Because X is a P-supermartingale, we have

$$\mathbb{E}^{\mathsf{P}} X_\infty \leq \mathbb{E}^{\mathsf{P}} X_t \leq 1$$

and

$$\mathbb{E}^{\mathsf{P}}\left(X_\infty\mathbb{1}_{\{X_t<k\}}\right) \leq \mathbb{E}^{\mathsf{P}}\left(X_t\mathbb{1}_{\{X_t<k\}}\right) = \mathsf{Q}(t < \tau^X, X_t < k) = \mathsf{Q}(X_t < k) \quad \forall\, k \geq 0.$$

For $t \to \infty$ by dominated convergence then

$$\mathbb{E}^{\mathsf{P}}\left(X_\infty \mathbb{1}_{\{X_\infty < k\}}\right) \leq \mathsf{Q}(X_\infty < k) = 0 \quad \forall\, k \geq 0.$$

This implies that $X_\infty \mathbb{1}_{\{X_\infty < k\}} = 0$ P-a.s. for all $k \geq 0$. Therefore, $\mathsf{P}(X_\infty \in \{0, \infty\}) = 1$. Since $\mathbb{E}^{\mathsf{P}}(X_\infty) \leq 1$, it follows that $\mathsf{P}(X_\infty = \infty) = 0$ and thus $X_\infty = 0$ P-almost surely. □

Until here we have only considered the behaviour of the local P-martingale X under Q. But how do other processes change their behaviour, when passing from P to Q? This question is of particular interest, since we want to apply our results to the pricing of options written on more than one underlying stock. Let us assume that besides X there exists another process Y on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$. For all $n \in \mathbb{N}$ we set $\tau_n^Y = \inf\{t \geq 0 : \ Y_t > n\} \wedge n$ and $\tau^Y = \lim_{n\to\infty} \tau_n^Y$. Note that in what follows we identify Y with the process $\widetilde{Y}$ defined above.

Lemma 2.5.2. *Let Y be a non-negative càdlàg local* P*-martingale. Then* $\mathsf{Q}(\tau^X \leq \tau^Y) = 1$.

Proof.

$$\mathsf{Q}(\tau^Y < \tau^X) = \lim_{n\to\infty} \mathsf{Q}(\tau^Y < \tau_n^X) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\left(X_{\tau_n^X} \mathbb{1}_{\{\tau^Y < \tau_n^X\}}\right) = 0.$$

□

Moreover, we introduce **condition (T)**: $\mathsf{Q}(\tau^X = \tau^Y < \infty) = 0$.

Clearly, (T) is always fulfilled if X is a true martingale. Moreover, condition (T) also holds, if X and Y are independent under P. Indeed in this case for every $n \in \mathbb{N}$,

$$\begin{aligned}
\mathsf{Q}(\tau^Y = \tau^X < n) &= \lim_{m\to\infty} \mathsf{Q}(\tau_m^Y < \tau^X < n) = \lim_{m\to\infty}\lim_{k\to\infty} \mathsf{Q}(\tau_m^Y < \tau_k^X < n) \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}}\left(X_{\tau_k^X} \mathbb{1}_{\{\tau_m^Y < \tau_k^X < n\}}\right) \leq \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}}\left(X_{\tau_k^X} \mathbb{1}_{\{\tau_m^Y < n\}}\right) \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}} X_{\tau_k^X} \cdot \mathsf{P}(\tau_m^Y < n) = \lim_{m\to\infty} \mathsf{P}(\tau_m^Y < n) = 0.
\end{aligned}$$

However, in general it is hard to check condition (T), since it requires some knowledge of the joint distribution of τ_n^X and τ_m^Y for n, m large.

If X and Y are assumed to be càdlàg processes under P, they are also almost surely càdlàg under Q before time τ^X because P and Q are equivalent on every $\mathcal{F}_{\tau_n^X}$. Furthermore, since $\frac{1}{X}$ is a Q-martingale, it does not explode and therefore $X_{t-} \neq 0$ and $X_t \neq 0$ Q-almost surely for all $t \geq 0$. Thus, the process $Z := \frac{Y}{X}$ does also have almost surely càdlàg paths before time τ^X. Since from time τ^X on everything is constant, the only crucial question is whether $Z = \frac{Y}{X}$ has a left-limit at τ^X.

Lemma 2.5.3. *Let Y be a non-negative local* P*-martingale. Then $Z_t := \left(\frac{Y_t}{X_t}\right)_{0\leq t<\tau^X}$ is a local martingale on $(\Omega, \mathcal{F}_{\tau^X-}, (\mathcal{F}_{t\wedge\tau^X-})_{t\geq 0}, \mathsf{Q})$. Furthermore, setting $Z_t := \widetilde{Z}_t$ and $X_t = \infty$ on $\{t \geq \tau^X\}$ is the unique way to define Z and X after time τ^X such that $\frac{1}{X}$ and Z remain non-negative càdlàg local martingales on $[0, \infty)$ for all possible extensions of the measure* Q *from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$.*

Proof. First, we show that $Z = \frac{Y}{X}$ is a local Q-martingale on $\bigcup_{n\in\mathbb{N}}[0, \tau_n^X]$ with localizing sequence $(\tau_n^Y \wedge \tau_n^X)_{n\in\mathbb{N}}$. Indeed, we have for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}\left(Z_{\tau_n^Y\wedge\tau_n^X}\middle|\mathcal{F}_t\right) &= \mathbb{E}^{\mathsf{Q}}\left(\frac{Y_{\tau_n^Y\wedge\tau_n^X}}{X_{\tau_n^Y\wedge\tau_n^X}}\middle|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\frac{Y_{\tau_n^Y\wedge\tau_n^X}}{X_{t\wedge\tau_n^Y\wedge\tau_n^X}}\middle|\mathcal{F}_t\right) = \frac{Y_{t\wedge\tau_n^Y\wedge\tau_n^X}}{X_{t\wedge\tau_n^Y\wedge\tau_n^X}}\\
&= Z_{t\wedge\tau_n^Y\wedge\tau_n^X}
\end{aligned}$$

and by Lemma 2.5.2 we know that $\tau_n^X \wedge \tau_n^Y \to \tau^X$ Q-almost surely. Since Z is a non-negative local supermartingale up to time τ^X, we can apply Fatou's lemma twice with $s \leq t$:

$$\begin{aligned}
\widetilde{Z}_s &= \liminf_{u\to\tau^X, u<\tau^X, u\in\mathbb{Q}} Z_{s\wedge u} = \liminf_{u\to\tau^X, u<\tau^X, u\in\mathbb{Q}} \lim_{n\to\infty} Z_{s\wedge u\wedge\tau_n^X\wedge\tau_n^Y}\\
&\geq \liminf_{u\to\tau^X, u<\tau^X, u\in\mathbb{Q}} \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(Z_{t\wedge u\wedge\tau_n^X\wedge\tau_n^Y}\middle|\mathcal{F}_s\right) \geq \liminf_{u\to\tau^X, u<\tau^X, u\in\mathbb{Q}} \mathbb{E}^{\mathsf{Q}}\left(Z_{t\wedge u}|\mathcal{F}_s\right)\\
&\geq \mathbb{E}^{\mathsf{Q}}\left(\liminf_{u\to\tau^X, u<\tau^X, u\in\mathbb{Q}} Z_{t\wedge u}\middle|\mathcal{F}_s\right) = \mathbb{E}^{\mathsf{Q}}(\widetilde{Z}_t|\mathcal{F}_s),
\end{aligned}$$

where the second inequality is due to the fact that $\mathbb{E}^{\mathsf{Q}}\left(Z_{t\wedge u\wedge\tau_n^X\wedge\tau_n^Y}\middle|\mathcal{F}_s\right) \geq \mathbb{E}^{\mathsf{Q}}\left(Z_{t\wedge u}|\mathcal{F}_s\right)$ by the supermartingale property. By the convergence theorem for positive supermartingales, we conclude that $\widetilde{Z}_{\tau^X-} = Z_{\tau^X-}$ exists Q-almost surely in $\mathbb{R}_+$. To see that $\widetilde{Z}$ is indeed a local martingale and not only a supermartingale, we show that $\widetilde{Z}^{\tau_n^Z}$ is a uniformly integrable martingale for all $n \in \mathbb{N}$, where $\tau_n^Z = \inf\{t \geq 0|\ Z_t > n\} \wedge n$. Since $\widetilde{Z}$ is a non-negative supermartingale, it is sufficient to prove that the expectation of $\widetilde{Z}^{\tau_n^Z}$ is constant:

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}\widetilde{Z}_{\tau_n^Z} &= \mathbb{E}^{\mathsf{Q}}\left(\widetilde{Z}_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z<\tau^X\}} + \widetilde{Z}_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}}\right)\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}\left(Z_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z<\tau_m^X\wedge\tau_m^Y\}}\right) + \mathbb{E}^{\mathsf{Q}}\left(\widetilde{Z}_{\tau^X-}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}}\right)\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}\left(Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau_n^Z<\tau_m^X\wedge\tau_m^Y\}}\right) + \mathbb{E}^{\mathsf{Q}}\left(\lim_{m\to\infty} Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}}\right)\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}Z_{\tau_m^X\wedge\tau_m^Y} - \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}\left(Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau^X>\tau_n^Z\geq\tau_m^X\wedge\tau_m^Y\}}\right) = Z_0.
\end{aligned}$$

To prove the uniqueness of the extension of Z for all possible extensions of Q to $\mathcal{F}$, define for all $n \in \mathbb{N}$, $\tau_n^Z = \inf\{t \geq 0 :\ \overline{Z}_t > n\}$, where $\overline{Z}$ is an arbitrary càdlàg extension of $(Z_t)_{t<\tau^X}$. Then $(\tau_n^Z)_{n\in\mathbb{N}}$ is a localizing sequence for $\overline{Z}$ for all possible extensions of Q. Fix one of these extensions and call it Q^0. We have

$$\mathbb{E}^{\mathsf{Q}^0}(\overline{Z}_t^{\tau_n^Z}|\mathcal{F}_s) = \overline{Z}_s^{\tau_n^Z}\ \forall\ n \in \mathbb{N}.$$

Now for fix $n \in \mathbb{N}$ define the new measure Q^n on $\mathcal{F}$ via

$$\frac{d\mathsf{Q}^n}{d\mathsf{Q}^0} = \frac{\overline{Z}_{\tau_n^Z}}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}.$$

Note that Q^n is also an extension of Q from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$. Furthermore for all $\varepsilon \geq 0$,

$$\overline{Z}_{\tau^X-}^{\tau_n^Z} = \mathbb{E}^{\mathsf{Q}^n}\left(\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z}\middle|\mathcal{F}_{\tau^X-}\right) = \mathbb{E}^{\mathsf{Q}^0}\left(\frac{\overline{Z}_{\tau_n^Z}}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}\cdot\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z}\middle|\mathcal{F}_{\tau^X-}\right) = \mathbb{E}^{\mathsf{Q}^0}\left(\frac{\left(\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z}\right)^2}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}\middle|\mathcal{F}_{\tau^X-}\right),$$

because $\overline{Z}^{\tau_n^Z}$ must also be a uniformly integrable martingale under Q^n. Therefore, $\overline{Z}^{\tau_n^Z}$ and $(\overline{Z}^{\tau_n^Z})^2$ are both Q^0-martingales after time τ^X-, which implies that $\overline{Z}_{\varepsilon+\tau^X} = \overline{Z}_{\tau^X-}$ for all $\varepsilon \geq 0$. Thus, $\overline{Z} \equiv \widetilde{Z}$ is uniquely determined. $\square$

As usual to simplify notation we will identify Z with the process $\widetilde{Z}$ in the following.

Remark 2.5.4.

- Note that if condition (T) is satisfied, then $Z_{\tau^X} = Z_{\tau^X-} = 0$ on $\{\tau^X < \infty\}$ Q-almost surely.
- Even though we proved that Z_{τ^X-} exists Q-a.s. and also X_{τ^X-} is well-defined, this does not allow us to infer any conclusions about the set $\{Y_{\tau^X-}$ exists in $\mathbb{R}_+\}$ in general.
- For our purposes it is sufficient that local Q-martingales are càdlàg almost everywhere, since we are only interested in pricing and do not deal with an uncountable number of processes. One should however have in mind that in order to have *everywhere* regular paths some kind of augmentation is needed, cf. Chapter 3.

Remark 2.5.5. If $\Omega = C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^2)$ is the path space introduced in Lemma 2.2.6, (X, Y) is the coordinate process, and $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is the canonical filtration generated by (X, Y), then under the assumptions of Lemma 2.5.3 we can extend Q to $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$ such that

$$\mathsf{Q}\left(\omega_1(t) = \infty,\ \omega_2(t) = \omega_2(\tau^X-)\ \forall\ t \geq \tau^X\right) = 1.$$

Lemma 2.5.6. *Let Y be a non-negative local P-martingale and set $Z := \frac{Y}{X}$.*

1. *If X is a P-martingale, then Z is a strict local Q-martingale if and only if Y is a strict local P-martingale.*
2. *Assume that X is a strict local P-martingale. Then:*
   (a) *If Y is a P-martingale, then Z is a Q-martingale and $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$.*
   (b) *If Z is a strict local Q-martingale or Z is a Q-martingale with $\mathsf{Q}(\tau^X < \infty, Z_{\tau^X} > 0) > 0$, then Y is a strict local P-martingale.*
   (c) *If Z is a Q-martingale and if condition (T) holds, then Y is a P-martingale.*
   (d) *If Y is a strict local P-martingale and if condition (T) holds, then Z is a strict local Q-martingale.*

Proof.

1. This is obvious, because Q and P are locally equivalent, if X is a true P-martingale.
2. First note that

$$\begin{aligned}
\mathbb{E}^{\mathsf{P}} Y_0 = \mathbb{E}^{\mathsf{Q}} Z_0 &\geq \mathbb{E}^{\mathsf{Q}} Z_t = \mathbb{E}^{\mathsf{Q}}\left(Z_t \mathbb{1}_{\{t<\tau^X\}}\right) + \mathbb{E}^{\mathsf{Q}}\left(Z_t \mathbb{1}_{\{t\geq\tau^X\}}\right) \\
&= \mathbb{E}^{\mathsf{Q}}\left(\frac{Y_t}{X_t} \mathbb{1}_{\{t<\tau^X\}}\right) + \mathbb{E}^{\mathsf{Q}}\left(Z_{\tau^X} \mathbb{1}_{\{t\geq\tau^X\}}\right) \\
&= \mathbb{E}^{\mathsf{P}} Y_t + \mathbb{E}^{\mathsf{Q}}\left(Z_{\tau^X} \mathbb{1}_{\{t\geq\tau^X\}}\right) \geq \mathbb{E}^{\mathsf{P}} Y_t.
\end{aligned}$$

(a) Since Y is a positive local P-martingale, we have:

$$\begin{aligned}
& Y \text{ is a true } \mathsf{P}\text{-martingale} \\
\Leftrightarrow\quad & \mathbb{E}^{\mathsf{P}} Y_t = \mathbb{E}^{\mathsf{P}} Y_0 \text{ for all } t \geq 0 \\
\Leftrightarrow\quad & \mathbb{E}^{\mathsf{Q}} Z_t = \mathbb{E}^{\mathsf{Q}} Z_0 \text{ for all } t \geq 0,\ \ Z_{\tau^X} \mathbb{1}_{\{\tau^X < \infty\}} = 0 \ \mathsf{Q}\text{-a.s.}
\end{aligned}$$

(b) follows from (a).

(c) If (T) holds, $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ Q-almost surely, cf. Remark 2.5.5. Therefore, since Z is a Q-martingale, the above inequality turns into an equality and Y is a true P-martingale.

(d) follows from (c).

□

Example 2.5.7. (Continuation of Example 2.3.1)
For the following example we work on the path space $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^2)$ with (X, Y) denoting the coordinate process and $(\mathcal{F}_t)_{t \geq 0}$ being the right-continuous augmentation of the canonical filtration generated by the coordinate process. Remember from example 2.3.1 that for $\sigma(x)$ locally bounded and bounded away from zero for $x > 0$, $\sigma(0) = 0$, the local P-martingale

$$dX_t = \sigma(X_t) dW_t, \quad X_0 = 1,$$

is strictly positive whenever

$$\int_0^1 \frac{x}{\sigma^2(x)} dx = \infty,$$

and under Q with $\left.\frac{d\mathsf{P}}{d\mathsf{Q}}\right|_{\mathcal{F}_t} = \frac{1}{X_t}$ the reciprocal process is a true martingale with decomposition

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2} dW_t^{\mathsf{Q}} = \overline{\sigma}\left(\frac{1}{X_t}\right) dW_t^{\mathsf{Q}}$$

for the Q-Brownian motion $W_t^{\mathsf{Q}} = W_t - \int_0^t \frac{\sigma(X_s)}{X_s} ds$ defined on the set $\{t < \tau^X\}$ and $\overline{\sigma}(y) := -y^2 \cdot \sigma\left(\frac{1}{y}\right)$.
Now let us assume that Y is also a local martingale under P with dynamics

$$dY_t = \gamma(Y_t) dB_t,$$

where γ fulfills the same assumptions as σ and B is another P-Brownian motion such that $\langle B, W \rangle_t = \rho t$. Then $\frac{Y}{X}$ is a Q-local martingale with decomposition

$$d\left(\frac{Y_t}{X_t}\right) = \frac{\gamma(Y_t)}{X_t} dB_t^{\mathsf{Q}} + Y_t \overline{\sigma}\left(\frac{1}{X_t}\right) dW_t^{\mathsf{Q}},$$

where B^{Q} is a Q-BM defined up to time τ^X such that $\langle B^{\mathsf{Q}}, W^{\mathsf{Q}} \rangle_t = \rho t$ on $\{t < \tau^X\}$.

2.6 Application to financial bubbles II: last passage time formulas

In Section 2.4 we have seen how one can determine the influence bubbles have on option pricing formulas through a decomposition of the option value into a "normal" term and a default term (cf. Theorems 2.4.4 and 2.4.10). However this approach only works well for options written on one underlying. It is rather difficult to give a universal way of how to determine the influence of asset price bubbles on the valuation of more complicated options and we will not do this here in all generality. Instead, we will do the analysis for a special example, the so called exchange option, which allows us to connect results about last passage times with the change of measure that was defined in Subsection 2.2.2.

Again we suppose that assumption **(S)** holds throughout the entire section. In addition we assume that there exists another strictly positive process Y on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which is also a local P-martingale. Furthermore, in the following we will assume that X and Y are *continuous*. As in Section 2.5 we define $Z := \frac{Y}{X}$, which is a local Q-martingale.

2.6.1 Exchange option

With the interpretation of X and Y as two stock price processes and assuming an interest rate of $r = 0$, we can define the price of a European exchange option with strike $K \in \mathbb{R}_+$ (also known as the ratio of notionals) and maturity $T \in \mathbb{R}_+$ as

$$E(K,T) := \mathbb{E}^{\mathsf{P}}(X_T - KY_T)^+.$$

The corresponding price of the American option is given by

$$A(K,T) := \sup_{\sigma \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}}(X_\sigma - KY_\sigma)^+,$$

where $\mathcal{T}_{0,T}$ is the set of all stopping times σ, which take values in $[0,T]$. Let us define the last passage time $\rho_K := \sup\left\{t \geq 0 \middle| \, Z_t = \frac{1}{K}\right\}$, where as usual the supremum of the empty set is equal to zero. In the next theorem the prices of the European and American exchange option are expressed in terms of the last passage time ρ_K in the spirit of [64].

Theorem 2.6.1. *For all $K, T \geq 0$ the prices of the European and American exchange option are given by*

$$E(K,T) = \mathbb{E}^{\mathsf{Q}}\left((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq T < \tau^X\}}\right), \quad A(K,T) = \mathbb{E}^{\mathsf{Q}}\left((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq T\}}\right).$$

Proof. Assume $\sigma \in \mathcal{T}_{0,T}$. As seen above, $Z = \frac{Y}{X}$ is a non-negative local Q-martingale, thus a supermartingale, which converges almost surely to $Z_\infty = Z_{\tau^X}$. From Corollary 3.4 in [11] resp. Theorem 2.5 in [64] we have the identity

$$\left(\frac{1}{K} - Z_\sigma\right)^+ = \mathbb{E}^{\mathsf{Q}}\left(\left(\frac{1}{K} - Z_{\tau^X}\right)^+ \mathbb{1}_{\{\rho_K \leq \sigma\}} \middle| \mathcal{F}_\sigma\right). \tag{2.6}$$

Multiplying the above equation with the $\mathcal{F}_\sigma$-measurable random variable $K\mathbb{1}_{\{\tau^X > \sigma\}}$ and taking expectations under Q yields

$$\mathbb{E}^{\mathsf{Q}}\left((1 - KZ_\sigma)^+ \mathbb{1}_{\{\tau^X > \sigma\}}\right) = \mathbb{E}^{\mathsf{Q}}\left((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq \sigma < \tau^X\}}\right).$$

Changing the measure via $d\mathsf{P}|_{\mathcal{F}_\sigma} = \frac{1}{X_\sigma} d\mathsf{Q}|_{\mathcal{F}_\sigma}$, we obtain

$$\begin{aligned} \mathbb{E}^{\mathsf{P}}\left(X_\sigma - KY_\sigma\right)^+ &= \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{\tau^X > \sigma\}} X_\sigma \left(1 - KZ_\sigma\right)^+\right) \\ &= \mathbb{E}^{\mathsf{Q}}\left((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq \sigma < \tau^X\}}\right), \end{aligned} \tag{2.7}$$

since $\mathbb{1}_{\{\tau^X > \sigma\}} = 1$ P-almost surely. Taking $\sigma = T$ the formula for the European option is proven.

For the American option value we note that in the proof of Theorem 1.4 in [8] it is shown that

$$A(K,T) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} \left(Y_{\tau_n^X \wedge T} \left(\frac{1}{Z_{\tau_n^X \wedge T}} - K \right)^+ \right) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} \left(X_{\tau_n^X \wedge T} - K Y_{\tau_n^X \wedge T} \right)^+ .$$

Setting $\sigma = \tau_n^X \wedge T$ in equality (2.7), it follows that

$$\begin{aligned} A(K,T) &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} \left(X_{\tau_n^X \wedge T} - K Y_{\tau_n^X \wedge T} \right)^+ = \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}} \left((1 - K Z_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau_n^X \wedge T < \tau^X\}} \right) \\ &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}} \left((1 - K Z_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau_n^X \wedge T\}} \right) = \mathbb{E}^{\mathsf{Q}} \left((1 - K Z_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau^X \wedge T\}} \right) \\ &= \mathbb{E}^{\mathsf{Q}} \left((1 - K Z_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le T\}} \right), \end{aligned}$$

where the last equality follows from the fact that $Z_{\tau^X} = \frac{1}{K}$ on $\{\rho_K > \tau^X\} = \{\rho_K = \infty\}$. □

Remark 2.6.2. Assume that $\mathsf{Q}(\tau^X < \infty) = 1$, i.e. $\mathbb{E}^{\mathsf{P}} X_t \stackrel{t\to\infty}{\longrightarrow} 0$. If we take $Y \equiv 1$ in the above theorem, we get the formula for the standard European call option expressed as a function of the last passage time of X as it can be found in [70] for the special case of Bessel processes or in [51]:

$$E(K,T) = \mathsf{Q} \left(\rho_K \le T < \tau^X \right). \tag{2.8}$$

More generally, for arbitrary Y formula (2.8) is still true, if (T) holds and $\mathsf{Q}(\tau^X < \infty) = 1$.

Remark 2.6.3. We can also express the price of a barrier exchange option in terms of the last passage time of Z at level $\frac{1}{K}$ as done in Theorem 2.6.1 for exchange options without barriers. For example, in the case of the Down-and-In exchange option we simply have to multiply equation (2.6) with the $\mathcal{F}_\sigma$-measurable random variable $\mathbb{1}_{\{\hat{m}_\sigma^X \le D\}}$.

We now analyze a few special cases of Theorem 2.6.1 in more detail:

1. **X is a true P-martingale**

   If X is a true P-martingale, the price process for X exhibits no asset price bubble. Then, regardless of whether the stock price process Y has an asset price bubble or not, we know that Q is locally equivalent to P and $\mathsf{Q}(\tau^X = \infty) = 1$. Therefore

   $$E(K,T) = A(K,T) = \mathbb{E}^{\mathsf{Q}} \left((1 - K Z_\infty)^+ \mathbb{1}_{\{\rho_K \le T\}} \right)$$

   and the European and American exchange option values are equal. For $Y \equiv 1$ this formula is well-known, cf. [64].

2. **Y is a true P-martingale**

   We recall from Lemma 2.5.6 that in this case $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ Q-almost surely. Denoting $\tau_0^Z = \inf\{t \ge 0 | \, Z_t = 0\}$ this translates into $\mathsf{Q}(\tau^X = \tau_0^Z) = 1$, since

   $$\mathsf{Q}(\tau_0^Z < \tau^X) = \lim_{n\to\infty} \mathsf{Q}(\tau_0^Z < \tau_n^X) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} \left(X_{\tau_n^X} \mathbb{1}_{\{\tau_0^Z < \tau_n^X\}} \right) = 0.$$

   Therefore,

   $$\begin{aligned} E(K,T) &= \mathsf{Q} \left(\rho_K \le T < \tau_0^Z \right), \\ A(K,T) &= \mathsf{Q} \left(\rho_K \le T \wedge \tau^X \right) = \mathsf{Q} \left(\rho_K \le T \wedge \tau_0^Z \right) = \mathsf{Q} \left(\rho_K \le T \right), \end{aligned}$$

where the last equality follows from the fact that the last passage time of the level $\frac{1}{K}$ by Z cannot be greater than its first hitting time of 0. Note that in this case the above formula for $E(K,T)$ is similar to the one for the European call option given in [51], Proposition 7, see also [70] for the case of the reciprocal Bessel process of dimension greater than two.
Especially, the American option premium is equal to

$$\begin{aligned} A(K,T) - E(K,T) &= \mathsf{Q}\left(\rho_K \leq T\right) - \mathsf{Q}\left(\rho_K \leq T < \tau_0^Z\right) = \mathsf{Q}\left(\rho_K \leq T, \tau_0^Z \leq T\right) \\ &= \mathsf{Q}(\tau_0^Z \leq T) = \mathsf{Q}(\tau^X \leq T) = \gamma_X(0,T), \end{aligned}$$

which is just the default of the local P-martingale X or, in other words, the bubble of the stock X between 0 and T.

3. **X and Y are both strict local P-martingales: An example**

Let X and Y be the reciprocals of two independent BES(3)-processes under P and assume that $X_0 = x \in \mathbb{R}_+$, while $Y_0 = 1$. (Note that this normalization is different from the previous one. However, since the density of X resp. Y is explicitly known in this case, we can do calculations directly under P. This allows us to point out some anomalies of the option value in the presence of strict local martingales.)

We apply the formula for the European call option value written on the reciprocal BES(3)-process from Example 3.6 in [15] and integrate over Y:

$$\begin{aligned} E(K,T) = \int_0^\infty x \left[\Phi\left(\frac{x - zK}{xzK\sqrt{T}}\right) - \Phi\left(-\frac{1}{x\sqrt{T}}\right) + \Phi\left(\frac{1}{x\sqrt{T}}\right) - \Phi\left(\frac{zK + x}{xzK\sqrt{T}}\right)\right] \mathsf{P}(Y_T \in dz) \\ -K \int_0^\infty z \left\{\Phi\left(\frac{zK + x}{xzK\sqrt{T}}\right) - \Phi\left(\frac{zK - x}{xzK\sqrt{T}}\right) + x\sqrt{T}\left[\varphi\left(\frac{zK + x}{xzK\sqrt{T}}\right) - \varphi\left(\frac{x - zK}{xzK\sqrt{T}}\right)\right]\right\} \mathsf{P}(Y_T \in dz), \end{aligned}$$

where

$$\mathsf{P}(Y_T \in dz) = \frac{1}{z^3} \frac{dz}{\sqrt{2\pi T}} \left(\exp\left(-\frac{(1/z - 1)^2}{2T}\right) - \exp\left(-\frac{(1/z + 1)^2}{2T}\right)\right).$$

Since $\mathbb{E}^{\mathsf{P}} X_T \stackrel{x\to\infty}{\longrightarrow} \frac{2}{\sqrt{2\pi T}}$ as shown in [32], the option value converges to a finite positive value as the initial stock price $X_0 = x$ goes to infinity. Therefore, the convexity of the payoff function does not carry over to the option value. This anomaly for stock price bubbles has been noticed before by e.g. [15, 32]. We refer for the economic intuition of this phenomenon to [32], where a detailed analysis of stock and bond price bubbles modelled by the reciprocal BES(3)-process is done.

Furthermore, recall that by Jensen's inequality the European exchange option value is increasing in maturity if X and Y are true martingales. However, in our example the option value is *not* increasing in maturity anymore: Indeed, because of $E(K,T) \leq \mathbb{E}^{\mathsf{P}} X_T \stackrel{T\to\infty}{\longrightarrow} 0$ the option value converges to zero as $T \to \infty$. Taking $Y \equiv 1$, this behaviour has been noticed before by e.g. [15, 32, 53, 60] and is also directly evident from the representation of $E(K,T)$ in Theorem 2.6.1.

2.6.2 Real-world pricing

Here we want to give another interpretation of Theorem 2.6.1. Note that from a mathematical point of view we have only assumed that X and Y are strictly positive local P-martingales for the result. Above we have interpreted P as the risk-neutral probability and X, Y as two stock price processes. Now note that we have the identity

$(X - KY)^+ = Y\left(\frac{1}{Z} - K\right)^+$. This motivates the following alternative financial setting: we take P to be the historical probability and assume that also $\mathsf{P}(Y_0 = 1) = 1$. Normalizing the interest rate to be equal to zero, the process $S := \frac{1}{Z}$ denotes the (discounted) stock price process, while Y is a candidate for the density of an equivalent local martingale measure (ELMM). Since Y and $X = YS$ are both strictly positive local P-martingales, they are P-supermartingales and cannot reach infinity under P. Thus, $S = \frac{1}{Z}$ is also strictly positive under P and does not attain infinity under P either.

As before X and Y are both allowed to be either strict local or true P-martingales. While the question of whether $X = YS$ is a true martingale or not is related to the existence of a stock price bubble as discussed earlier, the question of whether Y is a strict local martingale or not is connected to the absence of arbitrage. If Y is a uniformly integrable P-martingale, an ELMM for Z exists and the market satisfies NFLVR. However, as shown in [27] and explained in [8], even if Y is only a strict local martingale, a superhedging strategy for any contingent claim written on S exists. Therefore, the "normal" call option pricing formulas

$$E(K,T) = \mathbb{E}^{\mathsf{P}}\left(Y_T\left(S_T - K\right)^+\right), \quad A(K,T) = \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}}\left(Y_\sigma\left(S_\sigma - K\right)^+\right)$$

are still reasonable when Y is only a strict local martingale. This pricing method is also known as "real-world pricing", since we cannot work under an ELMM directly, but must define the option value under the real-world measure, cf. [63]. Note that if Y is a true martingale, we can define an ELMM P^* for S on $\mathcal{F}_T$ via $\mathsf{P}^*|_{\mathcal{F}_T} = Y_T.\mathsf{P}|_{\mathcal{F}_T}$ and the market satisfies the NFLVR property until time $T \in \mathbb{R}_+$. In this case we obtain the usual pricing formulas

$$E(K,T) = \mathbb{E}^{\mathsf{P}^*}(S_T - K)^+ \qquad \text{resp.} \qquad A(K,T) = \sup_{\sigma\in\mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}^*}(S_\sigma - K)^+.$$

Following [32] we can interpret the situation when Y is only a strict local martingale as the existence of a bond price bubble as opposed to the stock price bubble discussed above. This is motivated by the fact that the real-world price of a zero-coupon bond is strictly less than the (discounted) pay-off of one, if Y is a strict local martingale. Of course, it is possible to make a risk-free profit in this case via an admissible trading strategy. From Theorem 2.6.1 we have the following corollary:

Corollary 2.6.4. *For all $K, T \geq 0$ the values of the European and American call option under real-world pricing are given by*

$$E(K,T) = \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\rho_K^S \leq T < \tau^X\}}\right), \; A(K,T) = \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\rho_K^S \leq T\}}\right)$$

with $\rho_K^S = \sup\{t \geq 0 |\ S_t = K\}$.

From the above formulas for the European and American call options it can easily be seen that their values are generally different, unless $X = YS$ is a true P-martingale (in this case $\tau^X = \infty$ Q-a.s.). Therefore Merton's no early exercise theorem does not hold anymore, cf. also [8, 15, 38, 39].

Furthermore, note that we have the following formula for any bounded stopping time T:

$$E(K,T) = \mathbb{E}^{\mathsf{P}}(X_T - KY_T)^+ = \mathbb{E}^{\mathsf{Q}}\left(1 - KZ_T\right)^+ - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \leq T\}}(1 - KZ_T)^+\right),$$

where the second term equals $\mathsf{Q}(\tau^X \leq T)$, if (T) holds. For $Y \equiv 1$ this decomposition of the European call value is shown in [60].

Now we show that also the asymptotic behaviour of the European and American call option is unusual, when we allow X and / or Y to be strict local P-martingales. From the definition of the European call option value we easily see that

$$\lim_{K\to 0} E(K,T) = \mathbb{E}^{\mathsf{P}}(Y_T S_T) = \mathbb{E}^{\mathsf{P}} X_T = \mathsf{Q}(\tau^X > T), \qquad \lim_{K\to\infty} E(K,T) = 0.$$

Moreover, using the last passage time formula for the American call derived above, it follows that

$$\lim_{K\to 0} A(K,T) = \lim_{K\to 0} \mathsf{Q}(\rho_K^S \leq T) = 1,$$

since Z does not explode Q-a.s. and hence S is strictly positive under Q. Similarly, denoting $\rho^Z_{1/K} = \sup\left\{t \geq 0 |\ Z_t = \frac{1}{K}\right\}$, we get

$$\begin{aligned}\lim_{K\to\infty} A(K,T) &= \lim_{K\to\infty} \mathsf{Q}(\rho_K^S \leq T,\ S_{\tau^X} = \infty) = \lim_{K\to\infty} \mathsf{Q}(\rho^Z_{1/K} \leq T,\ Z_{\tau^X} = 0)\\ &= \mathsf{Q}(Z_{\tau^X} = Z_T = 0) = \mathsf{Q}(T \geq \tau^X, Z_{\tau^X} = 0),\end{aligned}$$

which may be strictly positive and equals $\mathsf{Q}(T \geq \tau^X) = \gamma_X(0,T)$ under (T). For the asymptotics in T we have

$$\begin{aligned}\lim_{T\to\infty} E(K,T) &= \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\tau^X = \infty\}}\right),\\ \lim_{T\to\infty} A(K,T) &= \mathbb{E}^{\mathsf{Q}}\left(1 - \frac{K}{S_{\tau^X}}\right)^+,\end{aligned}$$

and from the definition of the call option it is also clear that

$$\lim_{T\to 0} E(K,T) = \lim_{T\to 0} A(K,T) = (1-K)^+.$$

American option premium under real-world pricing

We keep the notation and interpretation introduced at the beginning of Subsection 2.6.2. However, we do not assume that Z and / or X are continuous anymore.

Lemma 2.6.5. *Let $h : \mathbb{R}_{++} \to \mathbb{R}_+$ be a Borel-measurable function s.t. $\lim_{x\to\infty} \frac{h(x)}{x} =: \eta$ exists in $\mathbb{R}_+$. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ via $g(x) = x \cdot h\left(\frac{1}{x}\right)$ for $x > 0$ and $g(0) = \eta$. We denote by $E(h,T) = \mathbb{E}^{\mathsf{P}}(Y_T h(S_T))$ the value of the European option with maturity T and payoff function h and by $A(h,T)$ the value of the corresponding American option. Then,*

$$E(h,T) = \mathbb{E}^{\mathsf{Q}} g(Z_T) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \leq T\}} g\left(Z_{\tau^X}\right)\right).$$

Furthermore, if in addition h is convex with $h(0) = 0$, $h(x) \leq x$ for all $x \in \mathbb{R}_+$ and $\eta = 1$, then

$$A(h,T) = \mathbb{E}^{\mathsf{Q}} g\left(Z_T\right).$$

Proof. For the European option value we have

$$E(h,T) = \mathbb{E}^{\mathsf{P}}(Y_T h(S_T)) = \mathbb{E}^{\mathsf{Q}}\left(g(Z_T)\mathbb{1}_{\{\tau^X > T\}}\right) = \mathbb{E}^{\mathsf{Q}} g(Z_T) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \leq T\}} g\left(Z_{\tau^X}\right)\right).$$

And for the American option value we get

$$\begin{aligned} A(h,T) &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\left(Y_{T\wedge\tau_n^X} h(S_{T\wedge\tau_n^X})\right) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(Z_{T\wedge\tau_n^X} h\left(\frac{1}{Z_{T\wedge\tau_n^X}}\right)\right) \\ &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}} g(Z_{T\wedge\tau_n^X}) = \mathbb{E}^{\mathsf{Q}} g(Z_{T\wedge\tau^X}) = \mathbb{E}^{\mathsf{Q}} g(Z_T), \end{aligned}$$

where the first equality is proven in [8] under the above stated assumptions on h and the fourth equality follows by dominated convergence since $g \leq 1$ is a bounded and continuous function. □

Under the assumptions of Lemma 2.6.5 the American option premium is thus equal to

$$A(h,T) - E(h,T) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \leq T\}} g\left(Z_{\tau^X}\right)\right).$$

Note that Lemma 2.6.5 is a generalization of Theorem A1 in [15]. Indeed, if Y is a uniformly integrable P-martingale (i.e. NFLVR is satisfied), $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ by part 2(a) of Lemma 2.5.6. Thus,

$$A(h,T) = E(h,T) + g(0)\cdot \mathsf{Q}\left(\tau^X \leq T\right) = E(h,T) + \gamma_X(0,T).$$

2.7 Multivariate strictly positive (strict) local martingales

So far the measure Q defined in Theorem 2.2.12 above is only associated with the local P-martingale X in the sense that $X_{\tau_n^X}.\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} = \mathsf{Q}|_{\mathcal{F}_{\tau_n^X}}$ for all $n \in \mathbb{N}$ and that $\frac{1}{X}$ is a true martingale under Q. One may now naturally wonder whether, given two (or more) positive local P-martingales X and Y, there exists a measure Q, under which $\frac{1}{X}$ and $\frac{1}{Y}$ are both true martingales. Obviously, this is the case, if X and Y are independent under P. In this section we will consider the case where X and Y are continuous local P-martingales, but not necessarily independent.

Theorem 2.7.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous augmentation of a standard system. Assume that X and Y are two strictly positive continuous local P-martingales with $d\langle X\rangle_t = f_t dt$, $d\langle Y\rangle_t = g_t dt$ and $d\langle X, Y\rangle_t = h_t dt$. Suppose that for all $t > 0$, the stochastic integral*

$$M_t = \int_0^t \frac{(f_sY_s - h_sX_s)g_s}{Y_sX_s(f_sg_s - h_s^2)} dX_s + \int_0^t \frac{(g_sX_s - h_sY_s)f_s}{Y_sX_s(f_sg_s - h_s^2)} dY_s$$

is well-defined. Denote by $\tau := \tau^X \wedge \tau^Y \wedge \tau^{\mathcal{E}}$ the minimum of the explosion times of the processes X, Y and $\mathcal{E}(M)$. Then there exists a measure Q on $\mathcal{F}_\infty$, under which $\frac{1}{\widetilde{X}}$ and $\frac{1}{\widetilde{Y}}$ defined via

$$\begin{aligned} \widetilde{X}_t &= X_t \mathbb{1}_{\{t<\tau\}} + \liminf_{s\to\tau, s<\tau, s\in\mathbb{Q}} X_s \mathbb{1}_{\{\tau\leq t<\infty\}}, \\ \widetilde{Y}_t &= Y_t \mathbb{1}_{\{t<\tau\}} + \liminf_{s\to\tau, s<\tau, s\in\mathbb{Q}} Y_s \mathbb{1}_{\{\tau\leq t<\infty\}} \end{aligned}$$

are both continuous non-negative local Q-martingales and $d\mathsf{P}|_{\mathcal{F}_t} = \frac{1}{\mathcal{E}(M)_t}\mathbb{1}_{\{t<\tau\}} d\mathsf{Q}|_{\mathcal{F}_t}$ for all $t \geq 0$.

Proof. The stochastic exponential $\mathcal{E}(M)$ is a continuous local P-martingale with localizing sequence

$$\tau_n := \inf\{t \geq 0 : \mathcal{E}(M)_t > n\} \wedge \inf\{t \geq 0 : X_t > n\} \wedge \inf\{t \geq 0 : Y_t > n\} \wedge n.$$

We define a consistent family of probability measures Q_n on $\mathcal{F}_{\tau_n}$ by

$$\left.\frac{d\mathsf{Q}_n}{d\mathsf{P}}\right|_{\mathcal{F}_{\tau_n}} = \mathcal{E}(M)_{\tau_n}, \quad n \in \mathbb{N}.$$

Using the same trick as in the proof of Theorem 2.2.12, we restrict each measure Q_n to $\mathcal{F}_{\tau_n -}$. Since $(\mathcal{F}_{\tau_n -})_{n\in\mathbb{N}}$ is a standard system by Lemma 2.2.7, there exists a unique measure Q on $\mathcal{F}_{\tau -}$, such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n}} = \mathsf{Q}_n$ for all $n \in \mathbb{N}$. For any stopping time S and $A \in \mathcal{F}_S$ we get

$$\mathsf{Q}(S < \tau_n, A) = \mathbb{E}^{\mathsf{P}}\left(\mathcal{E}(M)_{S\wedge\tau_n}\mathbb{1}_{\{S<\tau_n, A\}}\right) = \mathbb{E}^{\mathsf{P}}\left(\mathcal{E}(M)_S\mathbb{1}_{\{S<\tau_n,A\}}\right).$$

Taking $n \to \infty$ results in

$$\mathsf{Q}(S < \tau, A) = \mathbb{E}^{\mathsf{P}}\left(\mathcal{E}(M)_S\mathbb{1}_{\{S<\infty,A\}}\right).$$

It follows that P is locally absolutely continuous with respect to Q before τ. We choose an arbitrary extension of Q from $\mathcal{F}_{\tau-}$ to $\mathcal{F}_\infty$ as discussed on page 16. Next, according to Girsanov's theorem applied on $\mathcal{F}_{\tau_n}$,

$$\begin{aligned}
N_{t\wedge\tau_n} &:= X_t^{\tau_n} - \langle M^{\tau_n}, X^{\tau_n}\rangle_t = X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{(f_sY_s - h_sX_s)g_s}{Y_sX_s(f_sg_s - h_s^2)} d\langle X\rangle_s \\
&\quad - \int_0^{t\wedge\tau_n} \frac{(g_sX_s - h_sY_s)f_s}{Y_sX_s(f_sg_s - h_s^2)} d\langle X, Y\rangle_s \\
&= X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{(f_sY_s - h_sX_s)g_sf_s + (g_sX_s - h_sY_s)f_sh_s}{Y_sX_s(f_sg_s - h_s^2)} ds = X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{f_s}{X_s} ds
\end{aligned}$$

is a local Q-martingale. We apply Itô's formula:

$$\begin{aligned}
\frac{1}{X_{t\wedge\tau_n}} &= \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dX_s}{X_s^2} + \int_0^{t\wedge\tau_n} \frac{d\langle X\rangle_s}{X_s^3} \\
&= \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dN_s}{X_s^2} - \int_0^{t\wedge\tau_n} \frac{f_s}{X_s^3} ds + \int_0^{t\wedge\tau_n} \frac{f_s}{X_s^3} ds = \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dN_s}{X_s^2}.
\end{aligned}$$

Thus, $\frac{1}{X^{\tau_n}}$ is a local Q-martingale for all $n \in \mathbb{N}$. Since $\frac{1}{X}$ is continuous, $(\tau_m^{1/X})_{m\in\mathbb{N}}$ is a localizing sequence for $\frac{1}{X^{\tau_n}}$ on $(\Omega, \mathcal{F}_{\tau_n}, \mathsf{Q})$ for all $n \in \mathbb{N}$, where

$$\tau_m^{1/X} := \inf\left\{t \geq 0 : \frac{1}{X_t} > m\right\} \wedge m, \quad \tau^{1/X} := \lim_{m\to\infty} \tau_m^{1/X}.$$

Moreover, we have

$$\mathsf{Q}(\tau^{1/X} < \tau) = \lim_{n\to\infty} \mathsf{Q}(\tau^{1/X} < \tau_n) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\left(\mathcal{E}(M)_{\tau_n}\mathbb{1}_{\{\tau^{1/X}<\tau_n\}}\right) = 0,$$

because X is strictly positive under P. Since a process which is locally a local martingale is a local martingale itself, we conclude that $\frac{1}{X}$ is a positive local Q-martingale up to time τ with localizing sequence $(\tau_n \wedge \tau_n^{1/X})_{n\in\mathbb{N}}$. Especially, $\lim_{n\to\infty} X_{\tau_n} = \lim_{n\to\infty} X_{\tau_n\wedge\tau_n^{1/X}}$ exists Q-almost surely. Thus, $\frac{1}{\widetilde{X}}$ is a continuous positive Q-supermartingale and $\tau_n^{1/X} \to \infty$ Q-almost surely. Therefore,

$$1 \geq \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\widetilde{X}_{\tau_n^{1/X}}}\right) = \lim_{m\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\widetilde{X}_{\tau_n^{1/X}\wedge\tau_m}}\right) \geq \lim_{m\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\widetilde{X}_{\tau_m^{1/X}\wedge\tau_m}}\right) = 1,$$

where the two inequalities follow by the supermartingale property. Hence, $\frac{1}{\widetilde{X}}$ is a local Q-martingale.

For $\frac{1}{Y}$ the claim follows by analogous calculations. □

But are $\frac{1}{X}$ and $\frac{1}{Y}$ in the setting of Theorem 2.7.1 actually true Q-martingales or just local Q-martingales? If X (resp. Y) is a homogeneous diffusion, then we can show the following extension of the above theorem.

Lemma 2.7.2. *In the setting of Theorem 2.7.1 assume that X follows the P-dynamics*

$$dX_t = \sigma(X_t)dB_t$$

for some P-Brownian motion B, where $\sigma(\cdot)$ is locally bounded and bounded away from zero on $(0,\infty)$ and $\sigma(0) = 0$. Then $\frac{1}{X}$ is a Q-martingale, where the measure Q is constructed in Theorem 2.7.1.

Proof. Note that, with the notation used in the proof of Theorem 2.7.1, up to time τ the process N follows the dynamics

$$dN_t = \sigma(X_t)dB_t^{\mathsf{Q}},$$

where

$$B_t^{\mathsf{Q}} := B_t - \int_0^t \frac{\sigma(X_s)}{X_s}ds$$

is a Q-Brownian motion on $[0,\tau)$ by Lévy's theorem. Hence, the Q-dynamics of $\frac{1}{X}$ up to time τ are given by

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2}dB_t^{\mathsf{Q}} =: \overline{\sigma}\left(\frac{1}{X_t}\right)dB_t^{\mathsf{Q}} \tag{2.9}$$

and we are in a situation similar to Example 2.3.1. Especially, $\frac{1}{X}$ is a stopped homogeneous diffusion under Q. Recall that since X is strictly positive under P, we must have

$$\int_0^1 \frac{x}{\sigma^2(x)}dx = \infty.$$

But any diffusion on an auxiliary probability space starting from $X_0 = 1$ with the dynamics described in (2.9) satisfies

$$\int_1^\infty \frac{x}{\overline{\sigma}^2(x)}dx = \int_0^1 \frac{y}{\sigma^2(y)}dy = \infty$$

and is hence a true martingale by the criterion of [20], cf. Example 2.3.1. Naturally, any stopped diffusion with the same dynamics is a martingale as well. Since the fact whether $\frac{1}{X}$ is a true martingale or not only depends on its distributional properties, we may therefore conclude that $\frac{1}{X}$ is indeed a Q-martingale. □

Remark 2.7.3. Theorem 2.7.1 deals with two strictly positive local P-martingales. It is however obvious that one can get a similar result for $n \geq 2$ strictly positive local P-martingales. Also note that the construction in Theorem 2.7.1 is only possible if the local quadratic covariation matrix of the local P-martingales is sufficiently non-degenerate. Moreover, it is interesting that the statement of Lemma 2.7.2 contains no further restrictions on the stochastic behaviour of Y.

We briefly want to describe a different approach focusing on "conformal local martingales" in $\mathbb{R}^d$, $d > 2$, which is dealt with in [60].

Definition 2.7.4. A continuous local martingale X, taking values in $\mathbb{R}^d$, is called a conformal local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, if $\langle X^i, X^j\rangle = \langle X^1\rangle \mathbb{1}_{\{i=j\}}$ P-almost surely for all $1 \leq i, j \leq d$.

In [60] the authors make the restriction that the conformal local martingale does not enter some compact neighborhood of the origin in $\mathbb{R}^d$. Using simple localization arguments as in Theorem 2.2.12 above, one can get rid off this assumption which seems somehow inappropriate when dealing with stock price processes. This yields the following extended version of Lemma 12 in [60]. We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$.

Theorem 2.7.5. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space such that $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous augmentation of a standard system. For $d > 2$ let $X = (X^1, \dots, X^d)$ be a conformal local P-martingale. Suppose that $X_0 = x_0$ with $|x_0| = 1$ and define $\tau := \inf\{t \geq 0 |\ |X_t| = 0\}$. Then there exists a measure Q on $\mathcal{F}_\infty$, such that $\mathsf{Q}|_{\mathcal{F}_t} \gg \mathsf{P}|_{\mathcal{F}_t}$ for all $t \geq 0$ and such that*

$$Y_t := \begin{cases} \frac{X_t}{|X_t|^2} & : t < \tau \\ \liminf_{s\to\tau, s<\tau, s\in\mathbb{Q}} \frac{X_s}{|X_s|^2} & : t \geq \tau \end{cases}$$

is a conformal uniformly-integrable Q-martingale.

Proof. Note that $\mathsf{P}(\tau < \infty) = 0$ by Knight's theorem because a standard d-dimensional Brownian motion does not return to the origin almost surely for $d > 2$. We define the stopping times $\tau_n := \inf\{t \geq 0 :\ |X_t| \leq \frac{1}{n}\}$. As in Lemma 11 in [60] it follows that $\left(|X_{t\wedge\tau_n}|^{2-d}\right)_{t\geq 0}$ is a uniformly integrable P-martingale for all $n \in \mathbb{N}$, because $|\cdot|^{2-d}$ is harmonic. We define a consistent family of probability measures Q_n on $\mathcal{F}_{\tau_n}$ by

$$\left.\frac{d\mathsf{Q}_n}{d\mathsf{P}}\right|_{\mathcal{F}_{\tau_n}} = |X_{\tau_n}|^{2-d}, \quad n \in \mathbb{N}.$$

Using the same trick as in the proof of Theorem 2.2.12, we restrict each measure Q_n to $\mathcal{F}_{\tau_n -}$. Since $(\mathcal{F}_{\tau_n -})_{n\in\mathbb{N}}$ is a standard system, there exists a unique measure Q on $\mathcal{F}_{\tau-}$, such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n}} = \mathsf{Q}_n$ for all $n \in \mathbb{N}$. For any stopping time S we thus get

$$\mathsf{Q}(S < \tau_n) = \mathbb{E}^{\mathsf{P}}\left(|X_{\tau_n}|^{2-d}\mathbb{1}_{\{S<\tau_n\}}\right) = \mathbb{E}^{\mathsf{P}}\left(|X_S|^{2-d}\mathbb{1}_{\{S<\tau_n\}}\right).$$

Choosing $S = t < \infty, A \in \mathcal{F}_t$ and taking $n \to \infty$ results in

$$\mathsf{Q}(A \cap \{t < \tau\}) = \mathbb{E}^{\mathsf{P}}\left(|X_t|^{2-d}\mathbb{1}_A\right).$$

Therefore, P is locally absolutely continuous to Q before τ. As explained on page 16 there exists an extension of Q from $\mathcal{F}_{\tau-}$ to $\mathcal{F}_\infty$, which we also denote by Q.
From Lemma 12 in [60] we know that $\frac{X_{t\wedge\tau_n}}{|X_{t\wedge\tau_n}|^2}$ is a conformal Q_n-martingale. Furthermore,

$$\left(\mathbb{E}^{\mathsf{Q}} \sup_{t<\tau} |Y_t^i|\right)^2 \leq \mathbb{E}^{\mathsf{Q}} \sup_{t<\tau} |Y_t^i|^2 \leq 1, \quad 1 \leq i \leq d.$$

Thus Y is a continuous uniformly integrable Q-martingale by Exercise 1.48 in Chapter IV of [67]. Clearly, Y is also conformal. $\square$

Appendix

Condition (P)

In Theorem 2.2.1 we mentioned condition (P), which was introduced in Definition 4.1 in [57] following [61] as follows:

Definition 2..6. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0}$: $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We shall say that the property (P) holds if and only if $(\mathcal{F}_t)_{t\geq 0}$ enjoys the following conditions:

- For all $t \geq 0$, $\mathcal{F}_t$ is generated by a countable number of sets.
- For all $t \geq 0$, there exists a Polish space Ω_t, and a surjective map π_t from Ω to Ω_t, such that $\mathcal{F}_t$ is the σ-algebra of the inverse images by π_t of Borel sets in Ω_t, and such that for all $B \in \mathcal{F}_t$, $\omega \in \Omega$, $\pi_t(\omega) \in \pi_t(B)$ implies $\omega \in B$.
- If $(\omega_n)_{n\geq 0}$ is a sequence of elements of Ω such that for all $N \geq 0$,
$$\bigcap_{n\geq 0}^{N} A_n(\omega_n) \neq \emptyset,$$
where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then
$$\bigcap_{n\geq 0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

Chapter 3

A new kind of augmentation of filtrations suitable for a change of probability measure by a strict local martingale

In this chapter we introduce a new kind of augmentation of filtrations along a sequence of stopping times. This augmentation is suitable for the construction of a new probability measure associated with a positive strict local martingale as done in Chapter 2, while it is on the other hand rich enough to make classical results from stochastic analysis hold true on some stochastic interval of interest.

3.1 Introduction

The goal of this chapter is to introduce a new kind of augmentation of filtrations which is suitable for a change of probability measure associated with a strict local martingale. While it is safe and very convenient to work under the usual assumptions when doing a change of probability measure where the density process is a uniformly integrable martingale, one must be more careful if one takes a non-uniformly integrable martingale or a strict local martingale as a "potential" Radon-Nikodym density process.
Indeed it was already noted in Bichteler's book (2002) and later by Najnudel and Nikeghbali (2011) that in order to extend a consistent family of probability measures from $\bigcup_{t\geq 0}\mathcal{F}_t$ to $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$ one has to impose certain topological requirements on the probability space and one must refrain from the usual assumptions. This is however rather unsatisfactory in general, since results from stochastic analysis like the existence of regular versions of martingales do require some augmentation of the filtration. The existence of such versions is for example of interest whenever one considers an uncountable number of stochastic processes as it is often the case in dynamic optimization problems. This led Najnudel and Nikeghbali (2011) to introduce a new kind of augmentation of filtrations, the natural augmentation, that is compatible with the construction of a probability measure on $\mathcal{F}_\infty$ whose density process is defined via a non-uniformly integrable martingale.
While a positive strict local martingale (X_t), i.e. a local martingale which is not a true martingale, cannot directly serve as a Radon-Nikodym density process, it is still possible to construct a new measure Q on $\mathcal{F}_{\tau^X-}$ by extending the consistent family of measures

Q_n defined on $\mathcal{F}_{\tau_n^X}$ by

$$\mathsf{Q}_n = X_{\tau_n^X}.\mathsf{P}, \quad \tau_n^X = \inf\{t \geq 0 : \ X_t > n\} \wedge n, \quad \tau^X = \lim_{n\to\infty} \tau_n^X,$$

if the filtration on the underlying probability space is the right-augmentation of a so called standard system. Standard systems were introduced by Parthasarathy (1967) and first used in the above context by Föllmer (1972). Since in this case the measure Q is only uniquely defined on the sub-σ-algebra $\mathcal{F}_{\tau^X-}$ and is generally not absolutely continuous with respect to P on $\mathcal{F}_t$ for all $t \in \mathbb{R}$, we cannot use the natural augmentation introduced by Najnudel and Nikeghbali (2011). While the problem in Najnudel and Nikeghbali (2011) was the inclusion of null-sets from $\mathcal{F}_\infty$ in the initial filtration $\mathcal{F}_0$, the problem now becomes even more severe in that one can no longer include any null-events that happen after time τ^X in the initial filtration $\mathcal{F}_0$. This observation leads us to introduce a new kind of augmentation of filtrations along a sequence of stopping times that is on the one hand rich enough to make classical results from stochastic analysis hold true up to some stopping time and that on the other hand still allows for the construction of the new probability measure.
This chapter is organized as follows: in the next section we introduce a new kind of augmentation of filtrations along an increasing sequence of stopping times and we establish the existence of nice versions of stochastic processes up to some stopping time under the new augmentation. In section 3.3 we briefly review the construction of the above mentioned probability measure associated with a positive (strict) local martingale, before we apply the augmentation results from section 3.2 in this setting.

3.2 (τ_n)-natural assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space. We will start with a definition before stating the augmentation theorem.

Definition 3.2.1. Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of $(\mathcal{F}_t)$-stopping times.

- A subset $A \in \Omega$ is called $(\tau_n)_{n\in\mathbb{N}}$-negligible with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, iff there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of subsets of Ω s.t. for all $n \in \mathbb{N}$, $B_n \in \mathcal{F}_{\tau_n}$, $\mathsf{P}(B_n) = 0$, and $A \subset \bigcup_{n\in\mathbb{N}} B_n$.
- We say that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ is (τ_n)-complete, iff all the (τ_n)-negligible sets of Ω are contained in $\mathcal{F}_0$. It satisfies the (τ_n)-natural conditions, iff it is (τ_n)-complete and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.

Note that in the case of $\tau_n = n$, the above definition as well as the next theorem reduces to the case of the natural augmentation studied in [9] and [57], where all $\mathcal{F}_n^+$-negligible sets for all $n \in \mathbb{N}$ are included in $\mathcal{F}_0$. Thus, the following theorem can be seen as a generalization of Proposition 2.4 in [57].

Theorem 3.2.2. *Let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of stopping times on a filtered proability $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ and denote by $\mathcal{N}$ the family of all (τ_n)-negligible sets with respect to P. Set $\widetilde{\mathcal{F}} = \sigma(\mathcal{F}, \mathcal{N})$ and $\widetilde{\mathcal{F}}_t = \sigma(\mathcal{F}_t^+, \mathcal{N})$ for all $t \geq 0$. Then there exists a unique probability measure $\widetilde{\mathsf{P}}$ on $(\Omega, \widetilde{\mathcal{F}})$, which coincides with P on $\mathcal{F}$, and the space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ satisfies the (τ_n)-natural conditions. Moreover, $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ is the smallest extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which satisfies the (τ_n)-natural conditions. We therefore call it the (τ_n)-augmentation of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$.*

Proof. We only give a sketch of the proof, because all steps except the third one (which we do in detail) follow closely the proof of Proposition 2.4 in [57].

1. Define $\mathcal{E} = \{A \subset \Omega |\ \exists A' \in \mathcal{F} :\ A\Delta A' \in \mathcal{N}\}$. As in [57] it is easily checked that $\mathcal{E}$ is a σ-algebra and that $\mathcal{E} = \widetilde{\mathcal{F}}$. This implies that if $\widetilde{\mathsf{P}}$ is a probability on $(\Omega, \widetilde{\mathcal{F}})$ extending P we must have $\widetilde{\mathsf{P}}(A) = \mathsf{P}(A')$ for $A \in \widetilde{\mathcal{F}}$, where $A' \in \mathcal{F}$ satisfies $A\Delta A' \in \mathcal{N}$. Therefore, the measure $\widetilde{\mathsf{P}}$ is unique if it exists. Furthermore, $\widetilde{\mathcal{F}}_t = \{A \subset \Omega |\ \exists A' \in \mathcal{F}_t^+ :\ A\Delta A' \in \mathcal{N}\}$ for all $t \geq 0$ as can be easily verified.

2. Next we show that $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ is right-continuous:
For this assume that $A \in \bigcap_{s>t} \widetilde{\mathcal{F}}_s$. Therefore, $A \in \widetilde{\mathcal{F}}_{t+1/n}$ for all $n \in \mathbb{N}$ and there exists $A'_n \in \mathcal{F}_{(t+1/n)+}$ such that $A\Delta A'_n \in \mathcal{N}$ for all $n \in \mathbb{N}$. But then
$$A' := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A'_n \in \mathcal{F}_{t+}$$
and
$$A\Delta A' = A\Delta \left[\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A'_n\right] \in \mathcal{N},$$
which implies that $A \in \widetilde{\mathcal{F}}_t$.

3. The crucial step now is to show that for every $(\mathcal{F}_t^+)_{t\geq 0}$-stopping time T,
$$\widetilde{\mathcal{F}}_T = \sigma(\mathcal{F}_T^+, \mathcal{N}) = \{A \subset \Omega |\ \exists A' \in \mathcal{F}_T^+ :\ A\Delta A' \in \mathcal{N}\}.$$
Indeed it is well-known that T can be approximated from above by a sequence of simple stopping times. Because of the right-continuity of the filtration, it is therefore enough to show the claim for every simple $(\mathcal{F}_t^+)$-stopping time S. For this assume that S takes values in $\{t_1, \ldots, t_k, \infty\}$ with $0 \leq t_1 < t_2 < \cdots < t_k < \infty$. Then we have
$$\begin{aligned}
\widetilde{\mathcal{F}}_S &= \{A \in \widetilde{\mathcal{F}} \mid A \cap \{S \leq t\} \in \widetilde{\mathcal{F}}_t\ \forall\, t \geq 0\}\\
&= \{A \in \widetilde{\mathcal{F}} \mid A \cap \{S \leq t_l\} \in \widetilde{\mathcal{F}}_t\ \forall\, t \in [t_l, t_{l+1})\ \forall\, l = 1, \ldots, k\}\\
&= \{A \in \widetilde{\mathcal{F}} \mid A \cap \{S \leq t_l\} \in \widetilde{\mathcal{F}}_{t_l}\ \forall\, l = 1, \ldots, k\}\\
&= \widetilde{\mathcal{F}}_{S'} \cap \{A \in \widetilde{\mathcal{F}} \mid A \cap \{S \leq t_1\} \in \widetilde{\mathcal{F}}_{t_1}\}\\
&= \{A \in \widetilde{\mathcal{F}}_{S'} \mid A \cap \{S \leq t_1\} \in \widetilde{\mathcal{F}}_{t_1}\},
\end{aligned}$$
where $S' = S \vee t_2$. We will proceed by induction. Note that S' takes only the values $\{t_2, \ldots, t_k, \infty\}$ and by the induction hypothesis therefore
$$\widetilde{\mathcal{F}}_{S'} = \sigma(\mathcal{F}_{S'}^+, \mathcal{N}) = \{A \subset \Omega |\ \exists A' \in \mathcal{F}_{S'}^+ :\ A\Delta A' \in \mathcal{N}\}.$$
Let $A \in \widetilde{\mathcal{F}}_S$. Then $A \in \widetilde{\mathcal{F}}_{S'}$, which yields the existence of a set $A_0 \in \mathcal{F}_{S'}^+$ such that $A\Delta A_0 \in \mathcal{N}$ and $A_0 \cap \{S' \leq t_l\} = A_0 \cap \{S \leq t_l\} \in \mathcal{F}_{t_l}^+$ for all $l \in \{2, \ldots, k\}$. Furthermore, since $A \cap \{S \leq t_1\} \in \widetilde{\mathcal{F}}_{t_1}$, there exists some set $A_1 \in \mathcal{F}_{t_1}^+$ such that $A_1\Delta(A \cap \{S \leq t_1\}) \in \mathcal{N}$. Define $\overline{A} := (A_0 \cap \{S > t_1\}) \cup (A_1 \cap \{S \leq t_1\})$. Then $\overline{A} \in \mathcal{F}_S^+$:
$$\begin{aligned}
\overline{A} \cap \{S \leq t_1\} &= A_1 \cap \{S \leq t_1\} \in \mathcal{F}_{t_1}^+,\\
\overline{A} \cap \{S \leq t_l\} &= \underbrace{(A_1 \cap \{S \leq t_1\})}_{\in \mathcal{F}_{t_1}^+} \cup \left(\underbrace{\{S > t_1\}}_{\in \mathcal{F}_{t_1}^+} \cap \underbrace{(A_0 \cap \{S \leq t_l\})}_{\in \mathcal{F}_{t_l}^+}\right) \in \mathcal{F}_{t_l}^+\\
&\quad \forall\, l = 2, \ldots, k.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\overline{A}\Delta A &= [(A_0 \cap \{S > t_1\}) \cup (A_1 \cap \{S \leq t_1\})]\,\Delta A \\
&= ([(A_0 \cap \{S > t_1\}) \cup (A_1 \cap \{S \leq t_1\})]\setminus A) \\
&\quad \cup (A\setminus [(A_0 \cap \{S > t_1\}) \cup (A_1 \cap \{S \leq t_1\})]) \\
&\subset (A_0\setminus A) \cup [A_1\setminus(A \cap \{S \leq t_1\})] \cup [A\setminus(A_0 \cup A_1)] \cup [A\setminus(A_1 \cup \{S > t_1\})] \\
&\quad \cup [A\setminus(A_0 \cup \{S \leq t_1\})] \\
&\subset (A_0\setminus A) \cup [A_1\setminus(A \cap \{S \leq t_1\})] \cup [A\setminus(A_0 \cup A_1)] \cup [A\setminus(A_1 \cup \{S > t_1\})] \\
&\quad \cup [A\setminus(A_0 \cup \{S \leq t_1\})] \\
&\subset (A_0\setminus A) \cup [A_1\setminus(A \cap \{S \leq t_1\})] \cup [(A \cap \{S \leq t_1\})\setminus A_1)] \cup (A\setminus A_0) \\
&= (A\Delta A_0) \cup [A_1\Delta(A \cap \{S \leq t_1\})] \in \mathcal{N}.
\end{aligned}$$

Therefore, the claim follows by induction, once we show that it holds for the stopping time $S^* \in \{t_1, \infty\}$. For this note that

$$\widetilde{\mathcal{F}}_{S^*} \;=\; \{A \in \widetilde{\mathcal{F}} \mid A \cap \{S^* \leq t_1\} \in \widetilde{\mathcal{F}}_{t_1}\}.$$

Let $B \in \widetilde{\mathcal{F}}_{S^*}$. Then there exists $B_1 \in \mathcal{F}^+_{t_1}$ such that $B_1 \subset \{S^* \leq t_1\}$ and $B_1\Delta(B \cap \{S^* \leq t_1\}) \in \mathcal{N}$. Also, there exists $B_0 \in \mathcal{F}$ such that $B\Delta B_0 \in \mathcal{N}$. Now define $\overline{B} = B_1 \cup (B_0 \cap \{S^* > t_1\}) \in \mathcal{F}$. Then $\overline{B} \cap \{S^* \leq t_1\} = B_1 \in \mathcal{F}^+_{t_1}$ and

$$\begin{aligned}
\overline{B}\Delta B &= (B_1 \cup (B_0 \cap \{S^* > t_1\}))\Delta B \\
&= (B_1 \cup (B_0 \cap \{S^* > t_1\}))\Delta((B \cap \{S^* \leq t_1\}) \cup (B \cap \{S^* > t_1\})) \\
&\subset \underbrace{(B_1\Delta(B \cap \{S^* \leq t_1\})}_{\in\mathcal{N}} \cup (\underbrace{(B_0\Delta B)}_{\in\mathcal{N}} \cap \{S^* > t_1\}) \in \mathcal{N}.
\end{aligned}$$

Therefore, $\overline{B} \in \mathcal{F}^+_{S^*}$ and $\widetilde{\mathcal{F}}_{S^*} \subset \sigma(\mathcal{F}^+_{S^*}, \mathcal{N})$. The inclusion $\widetilde{\mathcal{F}}_{S^*} \supset \sigma(\mathcal{F}^+_{S^*}, \mathcal{N})$ is trivial. Finally, let T be an arbitrary $(\mathcal{F}^+_t)_{t\geq 0}$-stopping time and $(T_n)_{n\in\mathbb{N}}$ a decreasing sequence of simple stopping times such that $T_n \to T$ from above. Then, since the filtration is right-continuous,

$$\widetilde{\mathcal{F}}_T = \bigcap_{n\in\mathbb{N}} \widetilde{\mathcal{F}}_{T_n} = \bigcap_{n\in\mathbb{N}} \sigma(\mathcal{F}^+_{T_n}, \mathcal{N}) \supset \sigma(\mathcal{F}^+_T, \mathcal{N}).$$

To show the reverse inclusion take any set $A \in \widetilde{\mathcal{F}}_T$. From the above equality we get for each $n \in \mathbb{N}$ the existence of a set $A_n \in \mathcal{F}^+_{T_n}$ such that $A_n\Delta A \in \mathcal{N}$. We define $A'_n = \bigcup_{m\geq n} A_m \in \mathcal{F}^+_{T_n}$ for all $n \in \mathbb{N}$. Note that $(A'_n)_{n\in\mathbb{N}}$ is decreasing and that

$$\begin{aligned}
A'_n\Delta A &= \left(\bigcup_{m\geq n} A_m\right)\Delta A = \left(\bigcup_{m\geq n} A_m\setminus A\right) \cup \left(A\setminus \bigcup_{m\geq n} A_m\right) \\
&= \left(\bigcup_{m\geq n} \underbrace{A_m\setminus A}_{\in\mathcal{N}}\right) \cup \left(\bigcap_{m\geq n} \underbrace{A\setminus A_m}_{\in\mathcal{N}}\right) \in \mathcal{N}.
\end{aligned}$$

By the right-continuity of the filtration, we have

$$A' := \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m = \bigcap_{n\in\mathbb{N}} A'_n \in \mathcal{F}^+_T.$$

It remains to show that $A'\Delta A \in \mathcal{N}$. Indeed:

$$\begin{aligned} A'\Delta A &= \left(\bigcap_{n\in\mathbb{N}} A'_n\right)\Delta A = \left(\bigcap_{n\in\mathbb{N}} A'_n\backslash A\right) \cup \left(A\backslash \bigcap_{n\in\mathbb{N}} A'_n\right) \\ &= \left(\bigcap_{n\in\mathbb{N}} \underbrace{A'_n\backslash A}_{\in\mathcal{N}}\right) \cup \left(\bigcup_{n\in\mathbb{N}} \underbrace{A\backslash A'_n}_{\in\mathcal{N}}\right) \in \mathcal{N}. \end{aligned}$$

Therefore, $A \in \sigma(\mathcal{F}_T^+, \mathcal{N})$ and $\widetilde{\mathcal{F}}_T \subset \sigma(\mathcal{F}_T^+, \mathcal{N})$.

4. To show existence of the (τ_n)-augmentation we define for $A \in \widetilde{\mathcal{F}}$, $\widetilde{\mathsf{P}}(A) := \mathsf{P}(A')$, where $A' \in \mathcal{F}$ satisfies $A\Delta A' \in \mathcal{N}$. This definition does not depend on the particular choice of A'. Obviously, $\widetilde{\mathsf{P}}|_{\mathcal{F}} = \mathsf{P}$ and it is easily checked that $\widetilde{\mathsf{P}}$ is σ-additive. It remains to verify that $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ satisfies the (τ_n)-natural conditions: If $A \in \widetilde{\mathcal{F}}$ is (τ_n)-negligible, then there exist $(B_n)_{n\in\mathbb{N}}$ such that $B_n \in \widetilde{\mathcal{F}}_{\tau_n}$, $\widetilde{\mathsf{P}}(B_n) = 0$ for all $n \in \mathbb{N}$ and $A \subset \bigcup_{n\in\mathbb{N}} B_n$. Since $B_n \in \widetilde{\mathcal{F}}_{\tau_n}$, there exists $B'_n \in \mathcal{F}^+_{\tau_n}$ such that $B_n\Delta B'_n \in \mathcal{N}$. Thus, $\mathsf{P}(B'_n) = \widetilde{\mathsf{P}}(B_n) = 0$ and $B'_n \in \mathcal{N}$, which implies that also $B_n = (B'_n \cup (B_n\backslash B'_n))\backslash(B'_n\backslash B_n) \in \mathcal{N}$. It follows that $A \subset \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{N} \subset \widetilde{\mathcal{F}}_0$. Finally, it is easy to see that $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ is the smallest extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ that satisfies the (τ_n)-natural assumptions.

□

3.2.1 Martingales under the (τ_n)-natural augmentation

We have the following simple but important result which shows that martingale properties of stochastic processes are not changed when taking the (τ_n)-natural augmentation.

Lemma 3.2.3. (similar to Prop. 4.6 in [57]) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space and $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ its (τ_n)-augmentation with respect to an increasing sequence of $(\mathcal{F}_t)$-stopping times $(\tau_n)_{n\in\mathbb{N}}$. Let X be an $\mathcal{F}$-measurable P-integrable random variable. Then X is also integrable with respect to $\widetilde{\mathsf{P}}$ and $\mathbb{E}^{\widetilde{\mathsf{P}}}X = \mathbb{E}^{\mathsf{P}}X$. Moreover, $\mathbb{E}^{\widetilde{\mathsf{P}}}(X|\widetilde{\mathcal{F}}_t) = \mathbb{E}^{\mathsf{P}}(X|\mathcal{F}_t)$ $\widetilde{\mathsf{P}}$-a.s. for all $t \geq 0$.*

The proof is omitted, since it is exactly the same as the proof of Proposition 4.6 in [57].

Corollary 3.2.4. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space and let us denote by $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ its (τ_n)-augmentation, where $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times.*

1. *If $(X_t)_{t\geq 0}$ is an $\{(\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\}$-(super-)martingale, then it is also an $\left\{(\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}}\right\}$-(super)martingale.*

2. *If $(X_t)_{t\geq 0}$ is a local $\{(\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\}$-martingale, then it is also a local $\left\{(\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}}\right\}$-martingale.*

Proof. X is obviously $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$-adapted and by Lemma 3.2.3 X_t is integrable for all $t \geq 0$.

1. Furthermore, $\mathbb{E}^{\widetilde{\mathsf{P}}}(X_t|\widetilde{\mathcal{F}}_s) = \mathbb{E}^{\mathsf{P}}(X_t|\mathcal{F}_s) \overset{(\geq)}{=} X_s$ for all $s \leq t$ by Lemma 3.2.3.

2. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a localizing sequence for X under P. Since $(\sigma_n)_{n\in\mathbb{N}}$ are $(\mathcal{F}_t)_{t\geq 0}$-stopping times, $\{\sigma_n \leq t\} \in \mathcal{F}_t \subset \widetilde{\mathcal{F}}_t$ for all $t \geq 0$ and $\widetilde{\mathsf{P}}(\sigma_n \to \infty) = \mathsf{P}(\sigma_n \to \infty) = 1$, thus $(\sigma_n)_{n\in\mathbb{N}}$ is also a localizing sequence for X with respect to $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$. By 1. and Lemma 3.2.3 $(X_{t\wedge\sigma_n})_{t\geq 0}$ is a uniformly integrable $\left\{(\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}}\right\}$-martingale for all $n \in \mathbb{N}$.

□

In the following subsection we show that one can do even better: in fact, it is possible to construct for any martingale an adapted version with regular trajectories for all $\omega \in \Omega$ up to time $\tau = \lim_{n\to\infty} \tau_n$.

3.2.2 Existence of regular versions of trajectories up to time τ

As in [57] the following lemma, which relates the (τ_n)-natural conditions to the usual assumptions, is the main tool for establishing classical results from stochastic calculus under the (τ_n)-natural conditions.

Lemma 3.2.5. (similar to Prop. 2.5 in [57]) *Assume that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ satisfies the (τ_n)-natural assumptions for an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$. Then for all $n \in \mathbb{N}$ the space $(\Omega, \mathcal{F}_{\tau_n}, (\mathcal{F}_{t\wedge\tau_n})_{t\geq 0}, \mathsf{P})$ satisfies the usual assumptions.*

Proof. Let A be an $\mathcal{F}_{\tau_n}$-negligible set, i.e. there exists $B \in \mathcal{F}_{\tau_n}$ such that $A \subset B$ and $\mathsf{P}(B) = 0$. Thus, A is (τ_n)-negligible with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which is assumed to be (τ_n)-complete. Therefore, $A \in \mathcal{F}_0$. □

For the rest of this subsection let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space that satisfies the (τ_n)-natural assumptions for an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$. Denote $\tau = \lim_{n\to\infty} \tau_n$. Then on the subspace $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$ many classical results from stochastic analysis can be proven to be true in a similar way as it is done in section 3 of [57] under the natural assumptions. As an illustration of the usefulness of the (τ_n)-usual assumptions we now prove the existence of nice versions of martingales on $[0, \tau)$.

Theorem 3.2.6. (similar to Prop. 3.1 in [57]) *Let $(X_t)_{t\geq 0}$ be a supermartingale with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$. If $t \mapsto \mathbb{E}^{\mathsf{P}} X_{t\wedge\tau_n}$ is right-continuous for all $n \in \mathbb{N}$, then $(X_t)_{0\leq t<\tau}$ admits a càdlàg modification on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_{t\wedge\tau-})_{t\geq 0}, \mathsf{P})$, which is P-a.s. unique.*

Proof. Since the filtration is in particular assumed to be right-continuous, there exists a right-continuous adapted version of $(X_t)_{t\geq 0}$ by Lemma (1.1) in [29]. Denote it by $\overline{X}$. Then by Doob's optional sampling theorem the process $(\overline{X}_{t\wedge\tau_n})_{t\geq 0}$ is also a right-continuous supermartingale for every $n \in \mathbb{N}$, which is adapted to $(\mathcal{F}_{t\wedge\tau_n})_{t\geq 0}$. But the space $(\Omega, \mathcal{F}_{\tau_n}, (\mathcal{F}_{t\wedge\tau_n})_{t\geq 0}, \mathsf{P})$ satisfies the usual conditions by Lemma 3.2.5. Thus, $(\overline{X}_{t\wedge\tau_n})_{t\geq 0}$ admits a càdlàg modification, since $t \mapsto \mathbb{E}^{\mathsf{P}}(\overline{X}_{t\wedge\tau_n}) = \mathbb{E}^{\mathsf{P}}(X_{t\wedge\tau_n})$ is right-continuous. Let us denote this modification by $\widetilde{X}^{(n)}_t$, which is unique up to indistinguishability. Then $\widetilde{X}^{(n)}_t$ is also a modification of $(X_{t\wedge\tau_n})_{t\geq 0}$, and the uniqueness implies that the family $(\widetilde{X}^{(n)})_{n\in\mathbb{N}}$ is consistent, i.e. $\widetilde{X}^{(n+k)}_t \mathbb{1}_{\{t\leq\tau_n<\infty\}} = \widetilde{X}^{(n)}_t \mathbb{1}_{\{t\leq\tau_n<\infty\}}$ P-almost surely for all $t \geq 0$ and $n, k \in \mathbb{N}$. We define the set

$$N := \left\{\omega \in \Omega : \exists\, n, m \in \mathbb{N},\ n \geq m,\ \exists\, t \in [0, \tau_m] \cap \mathbb{R}_+ \text{ s.t. } \widetilde{X}^{(n)}_t(\omega) \neq \widetilde{X}^{(m)}_t(\omega)\right\},$$

which is (τ_n)-negligible. Therefore, $N \in \mathcal{F}_0$ and $\mathsf{P}(N) = 0$. Defining the process $(\widetilde{X}_t)_{0\leq t<\tau}$ on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$ via

$$\widetilde{X}_t(\omega) = \begin{cases} \widetilde{X}_t^{(n)}(\omega) & \text{, if } \omega \notin N \\ 0 & \text{, if } \omega \in N \end{cases}$$

for $t \in [0, \tau_n]$, we have constructed the desired càdlàg modification of $(X_t)_{0\leq t<\tau}$ on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$. □

Theorem 3.2.7. (similar to Prop. 3.3 in [57]) *Let $(X_t)_{t\geq 0}$ be an adapted process on the space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathsf{P})$ and assume that there exists a càdlàg version $(Y_t)_{0\leq t<\tau}$ of $(X_t)_{0\leq t<\tau}$, i.e. for all $t \geq 0$ s.t. $\mathsf{P}(\tau > t) > 0$ we have $\mathsf{P}(Y_t \neq X_t \mid t < \tau) = 0$. Then there exists a càdlàg and adapted version of $(X_t)_{0\leq t<\tau}$ on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_{t\wedge\tau-})_{t\geq 0}, \mathsf{P})$, which is indistinguishable from $(Y_t)_{0\leq t<\tau}$ on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$.*

Proof. We define the stopping times $(\tau_n^m)_{n,m\in\mathbb{N}}$ by

$$\tau_n^m := \sum_{k=1}^{\infty} \frac{k}{2^m} \mathbb{1}_{\left\{\frac{k-1}{2^m} \leq \tau_n < \frac{k}{2^m}\right\}}.$$

Then each τ_n^m takes only countably many values, $\tau_n^m \geq \tau_n^{m+1} \geq \tau_n$ and $\tau_n^m \to \tau_n$ as $m \to \infty$. Set

$$D = \left\{ \frac{k}{2^m} : \ k, m \in \mathbb{N} \right\},$$

which is a countable dense subset of $\mathbb{R}_+$. Furthermore, define the function $f_\omega : D \to \mathbb{R}$ via $f_\omega(t) = X_t(\omega)$. Then for all $n, m \in \mathbb{N}$ the set

$$N_{n,m} = \left\{\omega \in \Omega : \ f_\omega|_{[0,\tau_n^m(\omega)]\cap D} \text{ does not admit a unique càdlàg extension to } [0, \tau_n^m(\omega)]\right\}$$

is $\mathcal{F}_{\tau_n^m}$-measurable by Lemma 3.2 in [57], since $(X_{t\wedge\tau_n^m})_{t\geq 0}$ is adapted to $(\mathcal{F}_t \cap \mathcal{F}_{\tau_n^m})_{t\geq 0}$. Furthermore, $N_{n,m} \supset N_{n,m+1}$ for all $n, m \in \mathbb{N}$ and

$$N_n := \bigcap_{m\in\mathbb{N}} N_{n,m} \in \mathcal{F}_{\tau_n},$$

since the filtration is right-continuous. Because $(Y_t)_{0\leq t<\tau}$ is a càdlàg version of $(X_t)_{0\leq t<\tau}$, we must have

$$N_n \subset \{\omega \in \Omega \mid \exists\, t \in D \cap [0, \tau) : \ X_t(\omega) \neq Y_t(\omega)\} =: C$$

for all $n \in \mathbb{N}$. Since $(Y_t)_{0\leq t<\tau}$ is a version of $(X_t)_{0\leq t<\tau}$ on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$, $\mathsf{P}(C) = 0$ and therefore also $\mathsf{P}(N_n) = 0$ for all $n \in \mathbb{N}$, which implies that $N := \bigcup_{n\in\mathbb{N}} N_n$ is (τ_n)-negligible, i.e. $\mathsf{P}(N) = 0$ and $N \in \mathcal{F}_0$. Now, for $\omega \notin N$ let $g_{\omega,n}$ be the unique càdlàg extension of the function $f_{\omega,n} := f_\omega|_{[0,\tau_n)}$ from $D \cap [0, \tau_n)$ to $[0, \tau_n)$. By uniqueness the functions $(g_{\omega,n})_{n\in\mathbb{N}}$ are consistent, implying the existence of a càdlàg function $g_\omega : \ [0, \tau) \to \mathbb{R}$ such that $g_\omega(t) = X_t(\omega)$ for all $t \in D \cap [0, \tau)$. Next, we define the càdlàg process $(\overline{X}_t)_{0\leq t<\tau}$ by $\overline{X}_t(\omega) = g_\omega(t)\mathbb{1}_{\{\omega\notin N\}}$. Indeed, for all $t^\omega < \tau(\omega)$ and for every sequence $(t_n^\omega)_{n\in\mathbb{N}} \subset D \cap [0, \tau(\omega))$ tending to t^ω from above, we have

$$\overline{X}_{t^\omega}(\omega) = \mathbb{1}_{\{\omega\notin N\}} \lim_{n\to\infty} g_\omega(t_n^\omega) = \mathbb{1}_{\{\omega\notin N\}} \lim_{n\to\infty} X_{t_n^\omega}(\omega).$$

Because the filtration is right-continuous and $N \in \mathcal{F}_0$, the adaptedness of $(X_t)_{t\geq 0}$ implies that the process $(\overline{X}_t)_{0\leq t<\tau}$ is adapted to $(\mathcal{F}_{t\wedge\tau-})_{t\geq 0}$. Since Y is a version of X, $\overline{X}_t(\omega)$ is the right limit of Y at t restricted to $D \cap [0,\tau)$ for all $t \in [0,\tau)$ for almost all $\omega \in \Omega$. But since both, Y and $\overline{X}$, are càdlàg on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$, $\overline{X}_t = Y_t$ for $t \in [0,\tau)$ P-almost surely. Since $\mathsf{P}(Y_t \neq X_t \mid t < \tau) = 0$ for all $t \geq 0$ with $\mathsf{P}(\tau > t) > 0$ by assumption, $\overline{X}$ is a càdlàg and adapted version of X on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_{t\wedge\tau-})_{t\geq 0}, \mathsf{P})$. Moreover, since both, $\overline{X}$ and Y, are càdlàg versions of X on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_{t\wedge\tau-})_{t\geq 0}, \mathsf{P})$, they must be indistinguishable. □

It should be obvious that in a similar way other classical results of stochastic analysis like the Doob-Meyer decomposition or the existence of stochastic integrals can be proven up to time τ. We will not go in any more details here, but instead concentrate on the application of the (τ_n)-natural augmentation in the context of the extension of probability measures associated with strict local martingales in the next section.

3.3 Change of measure by a (strict) local martingale

Since in this chapter we use slightly different notation than in Subsection 2.2.2 of Chapter 2, we briefly review the construction of a probability measure associated with a positive (strict) local martingale as it was done in Subsection 2.2.2.

In the following let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space. Furthermore, we denote by $(\mathcal{F}_t^+)_{t\geq 0}$ the right-continuous augmentation of $(\mathcal{F}_t)_{t\geq 0}$, i.e. $\mathcal{F}_t^+ := \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$. Note that for now the filtration is *not* completed with any negligible sets of $\mathcal{F}$. We will use the notation $\mathcal{F}_{t\wedge\tau-}^+ := \mathcal{F}_t^+ \cap \mathcal{F}_{\tau-}^+$ for any $(\mathcal{F}_t^+)$-stopping time τ.

In order to be able to construct the measure Q associated with a (strict) local martingale X mentioned in the introduction, the filtration $(\mathcal{F}_t)$ has to be a standard system, cf. Definition 2.2.5. Recall that the most important examples of standard systems are the filtrations generated by the coordinate process on the spaces $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ or $D'(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ of all non-negative continuous resp. càdlàg functions $(\omega(t))_{t\geq 0}$ that have left limits on $(0, \alpha(\omega))$ for some $\alpha(\omega) \in [0,\infty]$ and remain constant on $[\alpha(\omega), \infty)$ at the value $\lim_{t\uparrow\alpha(\omega)} \omega(t)$ if this limit exists and at ∞ otherwise. Note that the spaces $C(\mathbb{R}_+, \mathbb{R})$ or $C([0,1], \mathbb{R})$, endowed with the filtrations generated by the coordinate process, are not standard systems. Adding the point $\{\infty\}$ is crucial.

In Subsection 2.2.2 of the previous chapter the following theorem was proven. It is a generalization of Theorem 4 in [16] and Proposition 1 in [60] to càdlàg local martingales on more general probability spaces and its proof relies on the construction of the Föllmer measure, cf. [29]. In Theorem 3.3.2 below we will state a further extension of this result involving the new kind of augmentation of filtrations introduced in Section 3.2.

Theorem 3.3.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space and assume that $(\mathcal{F}_t)_{t\geq 0}$ is a standard system. Let X be a càdlàg local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t^+)_{t\geq 0}, \mathsf{P})$ with values in $(0,\infty)$ and $X_0 = 1$ P-almost surely. We define the stopping times $\tau_n^X := \inf\{t \geq 0 : X_t > n\} \wedge n$ and $\tau^X = \lim_{n\to\infty} \tau_n^X$. Then there exists a unique probability measure Q on $\left(\Omega, \mathcal{F}_{\tau^X-}^+, (\mathcal{F}_{t\wedge\tau^X-}^+)_{t\geq 0}\right)$, such that $\frac{1}{X}$ is a Q-martingale up to time τ^X. Furthermore, $\mathsf{Q}|_{\mathcal{F}_t^+\cap\mathcal{F}_{\tau^X-}^+} \gg \mathsf{P}|_{\mathcal{F}_t^+\cap\mathcal{F}_{\tau^X-}^+}$ for all $t \geq 0$ with Radon-Nikodym derivative given by $\left.\frac{d\mathsf{P}}{d\mathsf{Q}}\right|_{\mathcal{F}_t^+\cap\mathcal{F}_{\tau^X-}^+} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}} = \frac{1}{X_t}$.*
Moreover, X is a strict local P-martingale if and only if $\mathsf{Q}(\tau^X < \infty) > 0$.

From here it is easy to see why we cannot work with the natural augmentation of $(\mathcal{F}_t)_{t\geq 0}$, but will have to use the (τ_n^X)-natural augmentation introduced in section 3.2. Indeed, we have $A_t := \{t \geq \tau^X\} \in \mathcal{F}_t^+ \cap \mathcal{F}_{\tau^X-}$ and $\mathsf{P}(A_t) = 0$ for all $t \geq 0$, while

$$\mathsf{Q}(A_t) = 1 - \mathsf{Q}(\tau^X > t) = 1 - \mathbb{E}^{\mathsf{P}}(X_t) > 0$$

for some $t > 0$, if X is a strict local martingale. Now, if $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ satisfied the natural conditions, then $A_t \in \mathcal{F}_0$ for all $t \geq 0$ and since $\mathsf{P}|_{\mathcal{F}_0} = \mathsf{Q}|_{\mathcal{F}_0}$ this would imply that $\mathsf{Q}(A_t) = \mathsf{P}(A_t) = 0$ for all $t \geq 0$, an obvious contradiction.

With the help of Section 3.2 we can nevertheless state the following extension of Theorem 3.3.1:

Theorem 3.3.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space and assume that $(\mathcal{F}_t)$ is a standard system. Let X be a càdlàg local martingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t^+)_{t\geq 0}, \mathsf{P})$ with values in $(0, \infty)$ and $X_0 = 1$ P-almost surely. We define the stopping times $\tau_n^X := \inf\{t \geq 0 : \ X_t > n\} \wedge n$, $\tau^X := \lim_{n\to\infty} \tau_n^X$ and denote by $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathsf{P}})$ the (τ_n^X)-augmentation of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$. Then there exists a unique probability measure $\widetilde{\mathsf{Q}}$ on $(\Omega, \widetilde{\mathcal{F}}_{\tau^X-}, (\widetilde{\mathcal{F}}_{t\wedge\tau^X-})_{t\geq 0})$, such that $\frac{1}{X}$ is a $\widetilde{\mathsf{Q}}$-martingale up to time τ^X. Furthermore, $\widetilde{\mathsf{Q}}|_{\widetilde{\mathcal{F}}_t \cap \widetilde{\mathcal{F}}_{\tau^X-}} \gg \widetilde{\mathsf{P}}|_{\widetilde{\mathcal{F}}_t \cap \widetilde{\mathcal{F}}_{\tau^X-}}$ for all $t \geq 0$ with Radon-Nikodym derivative given by $\left.\frac{d\widetilde{\mathsf{P}}}{d\widetilde{\mathsf{Q}}}\right|_{\widetilde{\mathcal{F}}_t \cap \widetilde{\mathcal{F}}_{\tau^X-}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$.*

Proof. Let $\left(\Omega, \overline{\mathcal{F}}_{\tau^X-}, (\overline{\mathcal{F}}_{t\wedge\tau^X-})_{t\geq 0}, \overline{\mathsf{Q}}\right)$ be the (τ_n^X)-augmentation of the filtered probability space $\left(\Omega, \mathcal{F}_{\tau^X-}^+, (\mathcal{F}_{t\wedge\tau^X-}^+)_{t\geq 0}, \mathsf{Q}\right)$ as constructed in Theorem 3.3.1. Then $\overline{\mathcal{F}}_{t\wedge\tau^X-} = \widetilde{\mathcal{F}}_{t\wedge\tau^X-}$ for $t \geq 0$ and $\widetilde{\mathcal{F}}_{\tau^X-} = \overline{\mathcal{F}}_{\tau^X-}$: A is (τ_n^X)-negligible with respect to $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^+)_{t\geq 0}, \mathsf{P}\right)$ iff there exist $(B_n)_{n\in\mathbb{N}}$ such that $A \subset \bigcup_{n\in\mathbb{N}} B_n$ and $B_n \in \mathcal{F}^+_{\tau_n^X}$, $\mathsf{P}(B_n) = 0$ for all $n \in \mathbb{N}$. Since $\mathsf{Q}|_{\mathcal{F}^+_{\tau_n^X}} \sim \mathsf{P}|_{\mathcal{F}^+_{\tau_n^X}}$, $\mathsf{Q}(B_n) = \mathsf{P}(B_n) = 0$. Thus, A is (τ_n^X)-negligible with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t^+)_{t\geq 0}, \mathsf{P})$ iff A is (τ_n^X)-negligible with respect to $\left(\Omega, \mathcal{F}_{\tau^X-}^+, (\mathcal{F}_{t\wedge\tau^X-}^+)_{t\geq 0}, \mathsf{Q}\right)$, i.e. $A \in \overline{\mathcal{F}}_{t\wedge\tau^X-}$ for $t \geq 0$.
Now let $A \in \widetilde{\mathcal{F}}_t$ for some $t \geq 0$, i.e. there exists $A' \in \mathcal{F}_t^+$ such that $A\Delta A'$ is (τ_n^X)-negligible with respect to Q and P. Then,

$$\widetilde{\mathsf{P}}(A) = \mathsf{P}(A') = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{A',\tau^X>t\}}\frac{1}{X_t}\right) = \mathbb{E}^{\overline{\mathsf{Q}}}\left(\mathbb{1}_{\{A',\tau^X>t\}}\frac{1}{X_t}\right) = \mathbb{E}^{\overline{\mathsf{Q}}}\left(\mathbb{1}_{\{A,\tau^X>t\}}\frac{1}{X_t}\right),$$

i.e. $\left.\frac{d\widetilde{\mathsf{P}}}{d\overline{\mathsf{Q}}}\right|_{\widetilde{\mathcal{F}}_t \cap \widetilde{\mathcal{F}}_{\tau^X-}} = \frac{1}{X_t}\mathbb{1}_{\{\tau^X>t\}}$. Identifying $\widetilde{\mathsf{Q}}$ with $\overline{\mathsf{Q}}$ yields the result. □

Let us briefly explain why the (τ_n^X)-natural augmentation is "good enough" for the setup considered here. First note that the measure Q is unevitably connected with the local martingale X. Therefore, it is not surprising that also the augmentation depends on the process X itself. On the other hand every process Y defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ is only defined up to time τ^X under Q. Since one is normally interested in the P-probability of events and uses the measure Q just as a helpful device to infer something about the P-probabilites, it is therefore almost always sufficient in applications to have results from stochastic analysis holding only until time τ^X, because everything that happens with positive probability under P takes place before time τ^X Q-almost surely.

Last but not least let us point out two situations in which it seems important to have nice versions of trajectories, i.e. processes which are regular everywhere and not only up to a nullset. Clearly, this is necessary if one considers an uncountable number of stochastic processes. As already mentioned in the introduction this happens regularly in optimization problems as for example in portfolio optimization. Indeed, even if the number of stocks is finite, the set of admissible trading strategies is in general so rich that the set of possible portfolio value processes is uncountable.

As a second example consider the occupation times formula, which requires in its proof a jointly continuous version of the field of local times at all levels and points in time. However, it was shown in [57] that without augmentation there does generally not exist a càdlàg and adapted version of the local time process at level $a \in \mathbb{R}$, i.e. local times can explode in finite time on some set. Hence, if one wants to apply powerful results from stochastic analysis like the occupation times formula, one should work with an augmented probability space.

Chapter 4

Change of measure up to a random time

In this chapter we extend results from Mortimer and Williams (1991) about changes of probability measure up to random times. Many new classes of examples involving honest times and pseudo-stopping times are provided. Furthermore, we discuss the question of no arbitrage up to a random time.

4.1 Introduction

Motivated by models from physics and chemistry Mortimer and Williams (1991) study how to perform a change of measure up to a random time $\sigma : (\Omega, \mathcal{F}) \to [0, \infty]$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$. More precisely, in their paper titled "Change of measure up to a random time: Theory" they derive the semimartingale decomposition of continuous $(\mathsf{P}, \mathcal{F}_t)$-martingales up to time σ in the progressively enlarged filtration

$$\mathcal{G}'_t = \mathcal{F}_t \vee \sigma \left(\mathbb{1}_{\{\sigma > s\}}; s \leq t \right)$$

under an equivalent probability measure Q and they give the expression of the $(\mathsf{Q}, \mathcal{G}'_t)$-hazard function of σ. To prove their results they use elementary methods and do no rely on the theory of enlargement of filtrations. Besides, Mortimer and Williams (1991) claim in their paper that "it is the *examples* which make this topic of some interest", but the only examples they provide deal with the well-known path decomposition of the standard Brownian motion.

In this chapter we extend their observations in numerous ways and point out their relevance for applications in mathematical finance. Working under the standing assumptions that σ avoids stopping times and that all $(\mathcal{F}_t)$-martingales are continuous, we are able to give more general examples involving honest times and pseudo-stopping times, especially we generalize the example of the Brownian path decomposition given in Mortimer and Williams (1991). While honest times are known to be well-suited for a progressive enlargement of filtration since the seminal work of Barlow (1978), pseudo-stopping times were only recently introduced by Nikeghbali and Yor (2005). As opposed to Mortimer and Williams (1991) who provide a Markovian study of their example our analysis is based on semimartingale calculus only.

As honest times are ends of optional sets their definition is independent of the underlying probability measure. This however is not true for pseudo-stopping times. We therefore investigate the question of whether there exist equivalent probability measures which leave the pseudo-stopping time property unchanged.

Furthermore, because the progressive enlargement of a filtration with an honest time ensures the stability of the semimartingale property also after time σ, we are able to extend the Girsanov-type theorem from Mortimer and Williams (1991) to the whole time horizon in this case. While the result itself is not very surprising and actually already known in greater generality, cf. [22], the way we prove it is interesting because as in Mortimer and Williams (1991) we solely use elementary methods and do not assume any prior knowledge of the theory of enlargements of filtrations. Actually, as it turns out there is a nice link to the so called relative martingales which were studied by Azéma, Meyer and Yor (1992).
Changes of measure are ubiquitous in mathematical finance. This is mainly due to the fundamental theorem of asset pricing which states in one form or the other that a market is free of arbitrage if and only if there exists an equivalent local martingale measure. A rigorous version of this statement involving the acronym NFLVR can be found in Delbaen and Schachermayer (1994). On the other hand, the technique of enlargements of filtrations is a standard method in mathematical finance to model credit risk and insider trading. This led us to the question of no arbitrage up to a random time σ: If we assume NFLVR with respect to the filtration $(\mathcal{F}_t)$, under which conditions does the market then also satisfy NFLVR with respect to $(\mathcal{G}_t)$ until time σ? This question is of great interest. Especially, it is known that honest times allow for arbitrage on the time horizon $[0, \sigma]$ in the progressively enlarged filtration. This was recently studied in detail by Fontana, Jeanblanc and Song (2013). We treat the general case here. Even though our results are not as complete as the ones of Fontana et al. (2013), we are able to give sufficient criteria for the validity of NFLVR on the time horizon $[0, \sigma]$ for general σ.
This chapter is organized as follows: In the next section we introduce the general setup and notation before we recall the result from [56] and give some first corollaries and slightly extended versions of their theorem. Applications to honest times and pseudo-stopping times can be found in Section 4.3. In Subsection 4.3.3 we generalize the example from [56]. In order to understand the relationship between the P-and Q-Azéma supermartingale we deal with their multiplicative decomposition in Section 4.4. Section 4.5 studies the stability of the pseudo-stopping time property with respect to certain measure changes. Financial applications can be found in Section 4.6 where we try to answer the question of no arbitrage up to a random time. Section 4.7 deals with locally absolutely continuous measure changes and in the last section we study changes of measure after time σ for honest times.

4.2 General theory

4.2.1 Setup and notation

Throughout the chapter we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, where $(\mathcal{F}_t)$ is assumed to satisfy the natural conditions, i.e. $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}_t$-negligible sets for all $t \in [0, \infty)$. By $\sigma : \Omega \to [0, \infty]$ we denote an $\mathcal{F}$-measurable random time, which gives rise to the progressively enlarged filtration

$$\mathcal{G}_t := \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\mathbb{1}_{\{\sigma>r\}}; r \leq s)\right).$$

For any $(\mathcal{G}_t)$-adapted process (X_t) we denote by $T_a^X = \inf\{t > 0 : \ X_t = a\}$ the first hitting time of the level $a \in \mathbb{R}$. If (X_t) is a real-valued stochastic process we denote by

$$\overline{X}_t := \sup_{s\leq t} X_s \qquad \text{and} \qquad \underline{X}_t := \inf_{s\leq t} X_s, \qquad t \geq 0,$$

its supremum resp. infimum process. Furthermore, $\mathcal{M}(\mathsf{P},\mathcal{F}_t)$ denotes the set of $(\mathsf{P},\mathcal{F}_t)$-martingales and $\mathcal{M}_{loc}(\mathsf{P},\mathcal{F}_t)$ resp. $\mathcal{M}_{u.i.}(\mathsf{P},\mathcal{F}_t)$ the set of local resp. uniformly integrable $(\mathsf{P},\mathcal{F}_t)$-martingales.

Throughout the chapter we will assume that the following two assumptions are satisfied:

(A) σ avoids any $(\mathcal{F}_t)$-stopping time: $\mathsf{P}(\sigma = T) = 0$ for any $(\mathcal{F}_t)$-stopping time T.
(C) All $(\mathcal{F}_t)$-martingales are continuous.

We denote by $Z_t^{\mathsf{P}} := \mathsf{P}(\sigma > t|\mathcal{F}_t)$ the Azéma supermartingale of σ. It decomposes as $Z_t^{\mathsf{P}} = m_t^{\mathsf{P}} - A_t^{\mathsf{P}}$ with $m_t^{\mathsf{P}} = \mathbb{E}^{\mathsf{P}}(A_\infty^{\mathsf{P}}|\mathcal{F}_t)$ being a uniformly integrable martingale and (A_t^{P}) being the $(\mathcal{F}_t)$-dual predictable projection of the process $(\mathbb{1}_{\{\sigma\leq t\}})_{t\geq 0}$. Under the assumptions (AC) the Azéma supermartingale is continuous and $Z_t^{\mathsf{P}} = m_t^{\mathsf{P}} - A_t^{\mathsf{P}}$ is thus its Doob-Meyer decomposition.

Let ρ be a non-negative $\mathcal{F}$-measurable random variable with expectation one. Then $\mathsf{Q} := \rho.\mathsf{P}$ defines a new probability measure which is absolutely continuous to P. We denote by (ρ_t) resp. $(\widetilde{\rho}_t)$ the optional projection of ρ on $(\mathcal{F}_t)$ resp. $(\mathcal{G}_t)$ satisfying for all $t \geq 0$,

$$\rho_t := \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{F}_t), \quad \widetilde{\rho}_t := \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{G}_t),$$

where $(\widetilde{\rho}_t)$ is chosen to be càdlàg and (ρ_t) is continuous due to (C). Furthermore, we define the $(\mathsf{P},\mathcal{F}_t)$-supermartingale

$$h_t := \mathbb{E}^{\mathsf{P}}(\rho\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t).$$

By Bayes' formula one has

$$h_t = \rho_t \cdot \mathsf{Q}(\sigma > t|\mathcal{F}_t) =: \rho_t Z_t^{\mathsf{Q}}.$$

Since σ avoids stopping times, $\mathsf{P}(\sigma = \infty) = 0$ and σ is finite P-almost surely. Therefore, Z^{P} and h both converge towards zero almost surely as $t \to \infty$.
If h is strictly positive, we denote by μ the stochastic logarithm of h, i.e. $h_t = \mathcal{E}(\mu)_t$. The process μ is again a $(\mathsf{P},\mathcal{F}_t)$-supermartingale with Doob-Meyer decomposition $\mu = \mu^L - \mu^F$, where $\mu^L \in \mathcal{M}_{loc}(\mathsf{P},\mathcal{F}_t)$ and μ^F is increasing. In general the process μ and hence also μ^L and μ^F are only well-defined on the stochastic interval $[0, T_0^h)$. Moreover, h, μ, μ^L, and μ^F are all continuous.

4.2.2 Girsanov-type theorems

We are now ready to recall the result of [56], Lemma 2.

Theorem 4.2.1. *Assume that h is strictly positive and let $U = (U_t)_{t\geq 0}$ be a local $(\mathsf{P},\mathcal{F}_t)$-martingale. Then the process $\left(\mathbb{1}_{\{\sigma>t\}} V_t \exp(\mu_t^F)\right)_{t\geq 0}$ is a local $(\mathsf{Q},\mathcal{G}_t)$-martingale, where $V := U - \langle U, \mu\rangle$.*
Moreover, the process $(\mu_{t\wedge\sigma}^F)_{t\geq 0}$ is the $(\mathsf{Q},\mathcal{G}_t)$-dual predictable projection of $(\mathbb{1}_{\{\sigma\leq t\}})_{t\geq 0}$.

Proof. The claim is proven in [56] for $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma\left(\mathbb{1}_{\{\sigma>s\}}; s \leq t\right)$ instead of $\mathcal{G}_t$ defined above. However, since $(\mathcal{G}'_t)$-martingales remain martingales with respect to the right-continuous augmentation $(\mathcal{G}_t)$ of $(\mathcal{G}_t)$, the claim follows easily. □

As an immediate consequence of the above result we deduce

Corollary 4.2.2. *Assume that h is strictly positive. If $U \in \mathcal{M}_{loc}(\mathsf{P}, \mathcal{F}_t)$, then*

$$V_{t\wedge\sigma} = U_{t\wedge\sigma} - \langle U, \mu\rangle_{t\wedge\sigma} \in \mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t).$$

Proof. Taking $U \equiv 1$ in Theorem 4.2.1 yields that

$$H_t := \mathbb{1}_{\{\sigma>t\}} \exp(\mu_t^F) \in \mathcal{M}(\mathsf{Q}, \mathcal{G}_t).$$

Since V is continuous, H and V are orthogonal to each other. Hence, their product is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale if and only if V is also a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale as long as $H_{t-} > 0$, i.e. on the interval $[0, \sigma]$. □

Remark 4.2.3. If we choose $\rho \equiv 1$ in the above corollary, we recover the well-known enlargement formula up to time σ: For any $M \in \mathcal{M}_{loc}(\mathsf{P}, \mathcal{F}_t)$ we have

$$M_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{d\langle M, Z^{\mathsf{P}}\rangle_s}{Z_s^{\mathsf{P}}} \in \mathcal{M}_{loc}(\mathsf{P}, \mathcal{G}_t).$$

Remark 4.2.4. In [56] the authors prove their result without applying any results from the theory of progressive enlargement of filtrations. Of course, Corollary 4.2.2 can also be proven by applying first Girsanov's theorem and afterwards the enlargement formula under Q. For so called honest times this is done in paragraph 81 of [22], where a more general version of the above result is proven without assuming the continuity of the processes involved.

Next we show that Theorem 4.2.1 also holds if h is not necessarily strictly positive.

Theorem 4.2.5. *If $U = (U_t)_{t\geq 0}$ is a local $(\mathsf{P}, \mathcal{F}_t)$-martingale, then $X_t := \mathbb{1}_{\{\sigma>t\}} V_t \exp(\mu_t^F)$ and $V_{t\wedge\sigma}$ are local $(\mathsf{Q}, \mathcal{G}_t)$-martingales, where $V_{t\wedge\sigma} = U_{t\wedge\sigma} - \langle U, \mu\rangle_{t\wedge\sigma}$.*

Proof. First we show that $\mathsf{Q}(\sigma < T_0^h) = 1$. For this note that $T_0^h = T_0^\rho \wedge T_0^{Z^{\mathsf{Q}}}$ because $h_t = \rho_t Z_t^{\mathsf{Q}}$. But we have

$$\mathsf{Q}\left(T_0^\rho < \infty\right) = \mathbb{E}^{\mathsf{P}}\left(\rho \mathbb{1}_{\{T_0^\rho<\infty\}}\right) = \mathbb{E}^{\mathsf{P}}\left(\rho_\infty \mathbb{1}_{\{T_0^\rho<\infty\}}\right) = \mathbb{E}^{\mathsf{P}}\left(0 \cdot \mathbb{1}_{\{T_0^\rho<\infty\}}\right) = 0.$$

Since σ avoids stopping times under P and Q is absolutely continuous to P, $\mathsf{Q}(\sigma = T_0^h) = 0$ and σ is also Q-almost surely finite. Hence,

$$\mathsf{Q}(\sigma \geq T_0^h) = \mathsf{Q}(\sigma > T_0^h) = \mathsf{Q}(\sigma > T_0^{Z^{\mathsf{Q}}}) = \mathbb{E}^{\mathsf{Q}} Z^{\mathsf{Q}}_{T_0^{Z^{\mathsf{Q}}}} = 0.$$

Especially, this means that X is Q-a.s. well-defined since μ is well-defined on the interval $[0, T_0^h)$. Second for every $n \in \mathbb{N}$ we write $U_t^n := U_{t\wedge T_{1/n}^h}$, $t \geq 0$. According to Theorem 4.2.1, the process $X_t^n := X_{t\wedge T_{1/n}^h}$ is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale for every $n \in \mathbb{N}$. Therefore, X is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale on the interval $[0, T_0^h) = \bigcup_{n\in\mathbb{N}}[0, T_{1/n}^h]$ and since $[0, T_0^h) \supset [0, \sigma)$ Q-almost surely, this implies that

$$X_t = \mathbb{1}_{\{\sigma>t\}} V_t \exp(\mu_t^F) \in \mathcal{M}(\mathsf{Q}, \mathcal{G}_t).$$

Finally, $(V_{t\wedge\sigma})$ is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale by the same reasoning as in the proof of Corollary 4.2.2. □

4.3 Special cases

In this section we specify the above setting further. Some of the examples are chosen having financial applications in mind, while others are motivated by purely mathematical considerations. The following well-known Lemma, cf. e.g. [22], will be very useful.

Lemma 4.3.1.

1. *If G is a $(\mathcal{G}_t)$-predictable process, then there exists an $(\mathcal{F}_t)$-predictable process F such that for all $t \geq 0$,*
$$G_t \mathbb{1}_{\{t \leq \sigma\}} = F_t \mathbb{1}_{\{t \leq \sigma\}}.$$
2. *If ξ is a P-integrable variable, then*
$$\mathbb{E}^{\mathsf{P}}(\xi \mathbb{1}_{\{\sigma > t\}} | \mathcal{G}_t) = \mathbb{1}_{\{\sigma > t\}} \frac{\mathbb{E}^{\mathsf{P}}(\xi \mathbb{1}_{\{\sigma > t\}} | \mathcal{F}_t)}{Z_t^{\mathsf{P}}}.$$
3. *If T is a $(\mathcal{G}_t)$-stopping time, then there exists an $(\mathcal{F}_t)$-stopping time S such that*
$$T \wedge \sigma = S \wedge \sigma.$$

4.3.1 The case of pseudo-stopping times

In this section we want to perform a change of measure up to a pseudo-stopping time. Pseudo-stopping times were introduced in [58] as follows:

Definition 4.3.2. A positive random variable $\sigma : (\Omega, \mathcal{F}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is called a $(\mathsf{P}, \mathcal{F}_t)$-pseudo-stopping time if for every bounded $(\mathsf{P}, \mathcal{F}_t)$-martingale M, $\mathbb{E}^{\mathsf{P}} M_\sigma = \mathbb{E}^{\mathsf{P}} M_0$.

In [58] it is shown that pseudo-stopping times can be characterized in many different ways, which we recall in

Theorem 4.3.3. *The following are equivalent:*

(1) σ is a $(\mathsf{P}, \mathcal{F}_t)$-pseudo stopping time.

(2) $A_\infty^{\mathsf{P}} \equiv 1$ almost surely.

(3) $A_\sigma^{\mathsf{P}} \sim \mathcal{U}[0,1]$

(4) For any local $(\mathsf{P}, \mathcal{F}_t)$-martingale $M = (M_t)_{t \geq 0}$, the process $(M_{t \wedge \sigma})_{t \geq 0}$ is a local $(\mathsf{P}, \mathcal{G}_t)$-martingale.

(5) $Z^{\mathsf{P}} = 1 - A^{\mathsf{P}}$ is a decreasing $(\mathcal{F}_t)$-predictable process.

Proof. The equivalence between (1), (2), (4) and (5) is shown in Theorem 1 of [58], while the implication (1)⇒(3) is a direct consequence of Proposition 2 of [58]. However, the relation (2) ⇔ (3) also follows immediately from the general relation between the Laplace transforms of A_σ^{P} and A_∞^{P}. Indeed, since (A_t^{P}) is the dual predictable projection of $(\mathbb{1}_{\{\sigma \leq t\}})$, we have

$$\lambda \cdot \mathbb{E}^{\mathsf{P}} \left(e^{-\lambda A_\sigma^{\mathsf{P}}} \right) = \lambda \cdot \mathbb{E}^{\mathsf{P}} \left(\int_0^\infty e^{-\lambda A_u^{\mathsf{P}}} dA_u^{\mathsf{P}} \right) = 1 - \mathbb{E}^{\mathsf{P}} \left(e^{-\lambda A_\infty^{\mathsf{P}}} \right), \quad \lambda > 0.$$

□

In the following two examples σ is assumed to be a $(\mathsf{P}, \mathcal{F}_t)$-pseudo-stopping time.

Example 4.3.4. In this example we make use of part (3) of Theorem 4.3.3. Since A^{P}_σ is uniformly distributed on $[0,1]$, we can choose $\rho = f(A^{\mathsf{P}}_\sigma)$ with $f > 0$ an integrable function such that $\int_0^1 f(x)dx = 1$. Since (A^{P}_t) is the dual optional projection of $(\mathbb{1}_{\{\sigma\leq t\}})$ we have for any $\mathcal{F}_t$-measurable random variable F_t,

$$\mathbb{E}^{\mathsf{P}}\left(f(A^{\mathsf{P}}_\sigma)\mathbb{1}_{\{\sigma>t\}}F_t\right) = \mathbb{E}^{\mathsf{P}}\left(\int_0^\infty f(A^{\mathsf{P}}_u)\mathbb{1}_{\{u>t\}}F_t dA^{\mathsf{P}}_u\right) = \mathbb{E}^{\mathsf{P}}\left(F_t\int_{A^{\mathsf{P}}_t}^{A^{\mathsf{P}}_\infty} f(x)dx\right),$$

which allows us to compute

$$\begin{aligned} h_t &= \mathbb{E}^{\mathsf{P}}\left(f(A^{\mathsf{P}}_\sigma)\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\int_t^\infty f(A^{\mathsf{P}}_u)dA^{\mathsf{P}}_u\bigg|\mathcal{F}_t\right) = \int_{A^{\mathsf{P}}_t}^1 f(x)dx, \\ dh_t &= -f(A^{\mathsf{P}}_t)dA^{\mathsf{P}}_t, \\ d\mu_t &= \frac{dh_t}{h_t} = \frac{-f(A^{\mathsf{P}}_t)dA^{\mathsf{P}}_t}{\int_{A^{\mathsf{P}}_t}^1 f(y)dy} = -d\mu^F_t. \end{aligned}$$

Therefore, for every continuous local $(\mathsf{P},\mathcal{F}_t)$-martingale U the process $(U_{t\wedge\sigma})_{t\geq 0}$ is a local $(\mathsf{Q},\mathcal{G}_t)$-martingale. Moreover, the dual predictable projection of $\mathbb{1}_{\{\sigma\leq t\}}$ with respect to $(\mathsf{Q},\mathcal{G}_t)$ is given by $\mu^F_{t\wedge\sigma} = -\log\left(\int_{A^{\mathsf{P}}_{t\wedge\sigma}}^1 f(y)dy\right)$. Note that this particular choice of ρ does not have any effect on continuous $(\mathcal{G}_t)$-martingales until time σ: $(U_{t\wedge\sigma})$ is a local $(\mathsf{P},\mathcal{G}_t)$-martingale and a local $(\mathsf{Q},\mathcal{G}_t)$-martingale. This generalizes Example 2 in [56].

Example 4.3.5. Let M be a strictly positive uniformly integrable $(\mathsf{P},\mathcal{F}_t)$-martingale starting from $M_0 = 1$. Then we may choose $\rho = M_\sigma$ since $\mathbb{E}^{\mathsf{P}}M_\sigma = \mathbb{E}^{\mathsf{P}}M_0 = 1$. We have

$$\begin{aligned} h_t &= \mathbb{E}^{\mathsf{P}}\left(M_\sigma\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\mathbb{E}^{\mathsf{P}}(M_\sigma|\mathcal{G}_t)\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(M_{\sigma\wedge t}\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) \\ &= M_t\cdot\mathsf{P}(\sigma>t|\mathcal{F}_t) = M_t Z^{\mathsf{P}}_t \\ dh_t &= M_t dZ^{\mathsf{P}}_t + Z^{\mathsf{P}}_t dM_t + d\langle M, Z^{\mathsf{P}}\rangle_t = -M_t dA^{\mathsf{P}}_t + (1-A^{\mathsf{P}}_t)dM_t \\ d\mu_t &= \frac{dh_t}{h_t} = \frac{-dA^{\mathsf{P}}_t}{1-A^{\mathsf{P}}_t} + \frac{dM_t}{M_t} = \frac{dM_t}{M_t} + d\log(1-A^{\mathsf{P}}_t). \end{aligned}$$

Thus, in this case the dual predictable projection of $\mathbb{1}_{\{\sigma\leq t\}}$ is equal to $\mu^F_t = -\log(1-A^{\mathsf{P}}_t)$ $= -\log(Z^{\mathsf{P}}_t)$. Applying Corollary 4.2.2 we see that given a continuous local $(\mathsf{P},\mathcal{F}_t)$-martingale U the process

$$V_{t\wedge\sigma} = U_{t\wedge\sigma} - \int_0^{t\wedge\sigma}\frac{d\langle M,U\rangle_s}{M_s}$$

is a local $(\mathsf{Q},\mathcal{G}_t)$-martingale.

Remark 4.3.6. Note that we cannot choose $\rho = M_\infty$ instead because in general we have $\mathbb{E}^{\mathsf{P}}(M_\infty|\mathcal{G}_t) \neq M_\sigma$ unless σ is a stopping time.

4.3.2 The case of honest times

Definition 4.3.7. A random time σ on $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathsf{P})$ is called honest if for any $t>0$, σ is equal to an $\mathcal{F}_t$-measurable random variable on $\{\sigma<t\}$.

Remark 4.3.8. Note that the definition of an honest time does not depend on the probability measure, while the definition of a pseudo-stopping time does. Indeed, it is shown in Proposition (5,1) of [42] that if σ is honest, then there exists an optional set Λ such that $\sigma(\omega) = \sup\{t : (t,\omega) \in \Lambda\}$ on $\{\sigma < \infty\}$. Since under assumption (C) the optional and predictable σ-field are equal, we may assume w.l.o.g. that the set Λ is predictable. Moreover, $\mathsf{P}(\sigma = \infty) = 0$ due to (A) and therefore σ is the end of a predictable set in our setup.

The following result from [59], Theorem 4.1, will be used frequently in the next sections.

Lemma 4.3.9. *For an honest time σ there exists a non-negative local $(\mathsf{P}, \mathcal{F}_t)$-martingale $(N_t^{\mathsf{P}})_{t\geq 0}$ with $N_0^{\mathsf{P}} = 1$ and $N_t^{\mathsf{P}} \to 0$ P-a.s. such that*

$$Z_t^{\mathsf{P}} = \mathsf{P}(\sigma > t|\mathcal{F}_t) = \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}.$$

Lemma 4.3.10. *Let σ be an honest time and denote by $Z_t^{\mathsf{P}} = N_t^{\mathsf{P}}/\overline{N}_t^{\mathsf{P}}$ the multiplicative decomposition of $Z_t^{\mathsf{P}} = \mathsf{P}(\sigma > t|\mathcal{F}_t)$ given in Lemma 4.3.9. Then for all $x > A_t^{\mathsf{P}}$,*

$$\mathsf{P}(A_\sigma^{\mathsf{P}} \in dx|\mathcal{F}_t) = N_t^{\mathsf{P}} e^{-x} dx.$$

Proof. From Lemma 2.1 in [59] we know that for $x > 0$,

$$\mathsf{P}\left(\sup_{s\geq t} N_s^{\mathsf{P}} > x \,\middle|\, \mathcal{F}_t\right) = \left(\frac{N_t^{\mathsf{P}}}{x}\right) \wedge 1.$$

It then follows from Lemma 4.3.9 that $A_t^{\mathsf{P}} = \log(\overline{N}_t^{\mathsf{P}})$ and $A_\sigma^{\mathsf{P}} = A_\infty^{\mathsf{P}}$. Thus,

$$\mathsf{P}(A_\sigma^{\mathsf{P}} > x|\mathcal{F}_t) = \mathsf{P}(A_\infty^{\mathsf{P}} > x|\mathcal{F}_t) = \mathsf{P}(\overline{N}_\infty^{\mathsf{P}} > e^x|\mathcal{F}_t) = \mathbb{1}_{\{\overline{N}_t^{\mathsf{P}} > e^x\}} + \mathbb{1}_{\{\overline{N}_t^{\mathsf{P}} \leq e^x\}} N_t^{\mathsf{P}} e^{-x}.$$

$\square$

Example 4.3.11. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function such that $\int_0^\infty f(x)e^{-x}dx = 1$.

$$\begin{aligned}
h_t &= \mathbb{E}^{\mathsf{P}}\left(f(A_\sigma^{\mathsf{P}})\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\int_t^\infty f(A_u^{\mathsf{P}})dA_u^{\mathsf{P}} \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\int_{A_t^{\mathsf{P}}}^{A_\infty^{\mathsf{P}}} f(x)dx \,\middle|\, \mathcal{F}_t\right) \\
&= \mathbb{E}^{\mathsf{P}}\left(\int_{A_t^{\mathsf{P}}}^{A_\sigma^{\mathsf{P}}} f(x)dx \,\middle|\, \mathcal{F}_t\right) = N_t^{\mathsf{P}} \int_{A_t^{\mathsf{P}}}^\infty \int_{A_t^{\mathsf{P}}}^y f(x)dx\; e^{-y}dy = N_t^{\mathsf{P}} \int_{A_t^{\mathsf{P}}}^\infty \int_x^\infty e^{-y}dy f(x)dx \\
&= N_t^{\mathsf{P}} \int_{A_t^{\mathsf{P}}}^\infty f(x)e^{-x}dx \\
dh_t &= \int_{A_t^{\mathsf{P}}}^\infty f(x)e^{-x}dx\; dN_t^{\mathsf{P}} - N_t^{\mathsf{P}} f(A_t^{\mathsf{P}})e^{-A_t^{\mathsf{P}}} dA_t^{\mathsf{P}} \\
d\mu_t &= \frac{dh_t}{h_t} = \frac{dN_t^{\mathsf{P}}}{N_t^{\mathsf{P}}} - \frac{f(A_t^{\mathsf{P}})e^{-A_t^{\mathsf{P}}} dA_t^{\mathsf{P}}}{\int_{A_t^{\mathsf{P}}}^\infty f(x)e^{-x}dx} = \frac{dN_t^{\mathsf{P}}}{N_t^{\mathsf{P}}} + d\log\left(\int_{A_t^{\mathsf{P}}}^\infty f(x)e^{-x}dx\right)
\end{aligned}$$

In this case the dual predictable projection of $\mathbb{1}_{\{\sigma\leq t\}}$ is given by $\mu_t^F = -\log\left(\int_{A_t^{\mathsf{P}}}^\infty f(x)e^{-x}dx\right)$. Applying Corollary 4.2.2 we see that given a continuous local $(\mathsf{P}, \mathcal{F}_t)$-martingale U the process

$$V_{t\wedge\sigma} = U_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{d\langle N^{\mathsf{P}}, U\rangle_s}{N_s^{\mathsf{P}}}$$

is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale. Therefore, as in Example 4.3.4 this particular choice of ρ allows continuous local $(\mathsf{P}, \mathcal{G}_t)$-martingales to stay local $(\mathsf{Q}, \mathcal{G}_t)$-martingales until time σ.

4.3.3 Generalization of Example 1 from [56]

Let σ be an honest time whose Azéma supermartingale with respect to P will be denoted by $Z_t^\sigma = \mathsf{P}(\sigma > t|\mathcal{F}_t)$ with Doob-Meyer decomposition $Z_t^\sigma = m_t^\sigma - A_t^\sigma$ (instead of $Z_t^{\mathsf{P}} = m_t^{\mathsf{P}} - A_t^{\mathsf{P}}$) in this subsection. It is shown in [58] that

$$\pi = \sup \left\{ t < \sigma : \; Z_t^\sigma = \inf_{u \le \sigma} Z_u^\sigma \right\}$$

is a P-pseudo-stopping time. From Proposition 5 in [58] we know that $\inf_{u\le\sigma} Z_u^\sigma$ is uniformly distributed and that the supermartingale $Z_t^\pi = \mathsf{P}(\pi > t|\mathcal{F}_t)$ equals $Z_t^\pi = \inf_{u\le t} Z_u^\sigma = \underline{Z}_t^\sigma$ for all $t \ge 0$. We define

$$\rho := f\left(1 - \inf_{u\le\sigma} Z_u^\sigma\right) = f\left(1 - Z_\pi^\sigma\right) = f\left(1 - Z_\pi^\pi\right) = f\left(A_\pi^\pi\right)$$

for some $f \in \mathcal{C}^1[0,1]$, $f > 0$ with $\int_0^1 f(x)dx = 1$, where A_t^π is the dual predictable projection of $\mathbb{1}_{\{\pi\le t\}}$. Then $\mathbb{E}^{\mathsf{P}}\rho = 1$ and we have

$$h_t = \mathbb{E}\left(\rho \mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}\left(f(A_\pi^\pi)\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t\right) = \mathbb{E}\left(f(A_\pi^\pi)\mathbb{1}_{\{\sigma>t\ge\pi\}}|\mathcal{F}_t\right) + \mathbb{E}\left(f(A_\pi^\pi)\mathbb{1}_{\{\pi>t\}}|\mathcal{F}_t\right).$$

The second term on the RHS has already been computed in Example 4.3.4 as

$$\mathbb{E}\left(f(A_\pi^\pi)\mathbb{1}_{\{\pi>t\}}|\mathcal{F}_t\right) = \int_{A_t^\pi}^1 f(x)dx.$$

Concerning the first term we have

$$\mathsf{P}(\sigma > t \ge \pi|\ \mathcal{F}_t) = \mathsf{P}(\sigma > t|\mathcal{F}_t) - \mathsf{P}(\pi > t|\mathcal{F}_t) = Z_t^\sigma - Z_t^\pi$$

and

$$\begin{aligned}
\mathbb{E}\left(f(A_\pi^\pi)\mathbb{1}_{\{\sigma>t\ge\pi\}}|\mathcal{F}_t\right) &= \mathbb{E}\left(f(1-\underline{Z}_\pi^\sigma)\mathbb{1}_{\{\sigma>t\ge\pi\}}|\mathcal{F}_t\right) = \mathbb{E}\left(f(1-\underline{Z}_t^\sigma)\mathbb{1}_{\{\sigma>t\ge\pi\}}|\mathcal{F}_t\right) \\
&= f(1-\underline{Z}_t^\sigma)\cdot(Z_t^\sigma - Z_t^\pi) = f(A_t^\pi)\cdot(Z_t^\sigma - Z_t^\pi).
\end{aligned}$$

Hence,

$$\begin{aligned}
h_t &= \int_{A_t^\pi}^1 f(x)dx + f(A_t^\pi)(Z_t^\sigma - Z_t^\pi) \\
dh_t &= -f(A_t^\pi)dA_t^\pi + f'(A_t^\pi)(Z_t^\sigma - Z_t^\pi)dA_t^\pi + f(A_t^\pi)(dZ_t^\sigma - dZ_t^\pi).
\end{aligned}$$

Since $1 - A_t^\pi = Z_t^\pi = \underline{Z}_t^\sigma$, we have $supp(dA_t^\pi) = \{Z_t^\pi = Z_t^\sigma\}$, which implies that

$$dh_t = f(A_t^\pi)(dZ_t^\sigma - dZ_t^\pi - dA_t^\pi) = f(A_t^\pi)dZ_t^\sigma.$$

Therefore,

$$d\mu_t = \frac{dh_t}{h_t} = \frac{f(A_t^\pi)dZ_t^\sigma}{\int_{A_t^\pi}^1 f(x)dx + f(A_t^\pi)(Z_t^\sigma - Z_t^\pi)}, \qquad d\mu_t^F = \frac{f(A_t^\pi)dA_t^\sigma}{\int_{A_t^\pi}^1 f(x)dx + f(A_t^\pi)A_t^\pi},$$

where we used that $supp(dA_t^\sigma) \subset \{Z_t^\sigma = 1\}$. Thus, given a continuous local $(\mathsf{P}, \mathcal{F}_t)$-martingale U the process

$$V_{t\wedge\sigma} = U_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{f(A_s^\pi)d\langle m^\sigma, U\rangle_s}{\int_{A_s^\pi}^1 f(x)dx + f(A_s^\pi)(Z_s^\sigma - Z_s^\pi)}$$

is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale by Corollary 4.2.2.

We briefly recall Example 1 from [56], which deals with the path decomposition of the Brownian motion, to see how it fits in the above framework.

Example 4.3.12. For a standard Brownian motion B one defines the random times

$$\sigma = \sup\{t < T_1^B : \; B_t = 0\}, \qquad \pi = \sup\{t < \sigma : \; B_t = \overline{B}_t\},$$

i.e. σ is the time of the last zero of B before it first hits one, and π is the last time at which B reaches its supremum before σ.

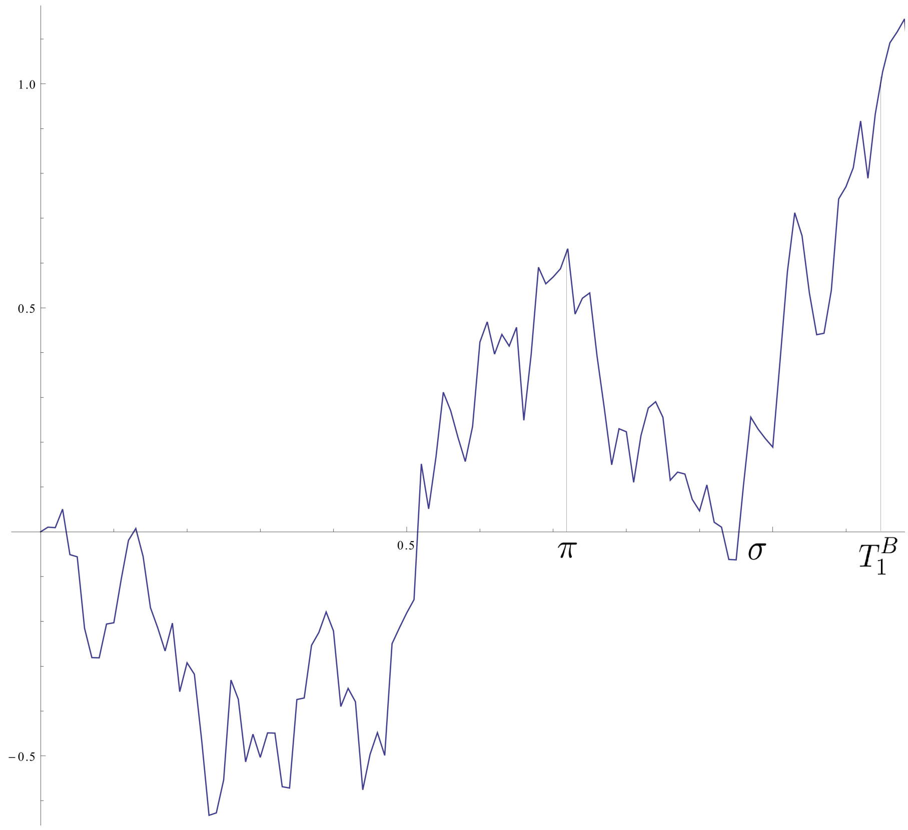

Figure 4.1: Williams' example of a pseudo-stopping time π.

Clearly, σ is an honest time and

$$Z_t^\sigma = \mathsf{P}(\sigma > t|\mathcal{F}_t) = 1 - B^+_{t\wedge T_1^B}.$$

Since

$$\pi = \sup\{t < \sigma : \; B_t = \overline{B}_t\} = \sup\{t < \sigma : \; Z_t^\sigma = \underline{Z}_t^\sigma\},$$

π is a pseudo-stopping time and $Z_t^\pi = 1 - \overline{B}_{t\wedge T_1^B}$, cf. Proposition 5 in [58]. In this case $A_\pi^\pi = \overline{B}_\sigma$ and

$$\begin{aligned} h_t &= \int_{\overline{B}_{t\wedge T_1^B}}^1 f(y)dy + f\left(\overline{B}_{t\wedge T_1^B}\right)\left(\overline{B}_{t\wedge T_1^B} - B^+_{t\wedge T_1^B}\right) \\ dh_t &= -f\left(\overline{B}_t\right)\left(\mathbb{1}_{\{B_t>0\}} dB_{t\wedge T_1^B} + \frac{dL_{t\wedge T_1^B}}{2}\right), \end{aligned}$$

where L denotes the local time of B at level zero. Hence, up to time σ the $(\mathsf{P}, \mathcal{F}_t)$-Brownian motion B follows the dynamics

$$dB_t = dW_t - \frac{\mathbb{1}_{\{B_t>0\}} f(\overline{B}_t)dt}{\int_{\overline{B}_t}^1 f(y)dy + f(\overline{B}_t)(\overline{B}_t - B_t^+)},$$

where (W_t) is a $(\mathsf{Q}, \mathcal{G}_t)$-Brownian motion. Especially, if we choose $f \equiv 1$, we see that B behaves like a reflected Brownian motion until time σ. This result is part of the well-known path decomposition of the standard Brownian motion due to Williams.

4.4 Multiplicative decompositions

When performing a change of measure up to a random time one needs to compute the process

$$h_t = \mathbb{E}^{\mathsf{P}}\left(\rho \mathbb{1}_{\{\sigma>t\}} | \mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\widetilde{\rho}_t \mathbb{1}_{\{\sigma>t\}} | \mathcal{F}_t\right),$$

where $\widetilde{\rho}_t$ was defined in Subsection 4.2.1. Therefore, the behaviour of the process $(\widetilde{\rho}_t)$ before time σ is of particular interest. In this section we will repeatedly make use of the following result, which is an immediate consequence of Théorème 5.12 and Lemme 5.15 of [42], cf. also Theorem 3.1 in [41].

Theorem 4.4.1. *For any bounded $\zeta \in \mathcal{G}_\sigma$ there exists a local $(\mathsf{P}, \mathcal{F}_t)$-martingale M and a bounded $(\mathcal{F}_t)$-predictable process K such that*

$$\mathbb{E}^{\mathsf{P}}(\zeta|\mathcal{G}_t) = M_t - \int_0^t \frac{d\langle M, Z^{\mathsf{P}}\rangle_s}{Z_s^{\mathsf{P}}} - \int_0^t \frac{K_s}{Z_s^{\mathsf{P}}} dA_s^{\mathsf{P}} \quad \text{on } \{\sigma > t\}.$$

Furthermore, if $t \mapsto \mathbb{E}^{\mathsf{P}}(\zeta|\mathcal{G}_t)$ is continuous almost surely (i.e. it does not jump at σ), then $K \equiv 0$.

Proof. To prove the theorem one can do exactly the same computations as in the proof of Theorem 3.1 in [41] without using any martingale representation property. Since we are only interested in the behaviour before time σ, we do not need the $(\mathcal{H}')$ hypothesis. □

Remark 4.4.2. The assumption (AC) is not needed to obtain a characterization of any bounded $(\mathcal{G}_t)$-martingale before time σ, cf. [41]. The above formulation is however sufficient for our purposes.

We have the following corollary which follows by localization and should be compared to Proposition 5.4 of [7].

Corollary 4.4.3. *Let $(\widetilde{\rho}_t)_{t\geq 0}$ be a non-negative continuous local $(\mathsf{P}, \mathcal{G}_t)$-martingale. Then there exists a local $(\mathsf{P}, \mathcal{F}_t)$-martingale M such that*

$$\widetilde{\rho}_{t\wedge\sigma} = M_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{d\langle M, Z^{\mathsf{P}}\rangle_s}{Z_s^{\mathsf{P}}}.$$

In the following we will repeatedly use the so called Itô-Watanabe decomposition of the Azéma supermartingale. Since it is less known than the additive Doob-Meyer decomposition, we briefly recall a continuous version of the result from [34], cf. also [5].

Theorem 4.4.4. *Let Z be a continuous non-negative supermartingale with Doob-Meyer decomposition $Z = m - A$. Then Z factorises uniquely as $Z = DN$, where N is a continuous non-negative local martingale starting from $N_0 = 1$ and D is a continuous decreasing process such that both N and D are constant on the set $\{Z = 0\}$. Moreover, N and D are given by*

$$D_t = Z_0 \exp\left(-\int_0^{t\wedge T_0^Z} \frac{dA_s}{Z_s}\right), \quad N_t = \mathcal{E}\left(\int_0^{t\wedge T_0^Z} \frac{dm_s}{Z_s}\right).$$

Remark 4.4.5. If $Z = Z^{\mathsf{P}}$ is the Azéma supermartingale of σ, then

$$Z_t^{\mathsf{P}} = 0 \quad \Leftrightarrow \quad m_t^{\mathsf{P}} - A_t^{\mathsf{P}} = \mathbb{E}^{\mathsf{P}}(A_\infty^{\mathsf{P}} - A_t^{\mathsf{P}}|\mathcal{F}_t) = 0 \quad \Leftrightarrow \quad A_t^{\mathsf{P}} = A_s^{\mathsf{P}} \ \forall\ s \geq t,$$

since A^{P} is an increasing process. Therefore, A^{P} and m^{P} only move on the set $\{Z^{\mathsf{P}} > 0\}$. Hence, in this case the processes

$$D_t^{\mathsf{P}} = \exp\left(-\int_0^t \frac{dA_s^{\mathsf{P}}}{Z_s^{\mathsf{P}}}\right), \quad N_t^{\mathsf{P}} = \mathcal{E}\left(\int_0^t \frac{dm_s^{\mathsf{P}}}{Z_s^{\mathsf{P}}}\right)$$

are well-defined and fulfill $supp(dD^{\mathsf{P}}) \subset \{Z^{\mathsf{P}} > 0\}$ resp. $supp(d\langle N^{\mathsf{P}}\rangle) \subset \{Z^{\mathsf{P}} > 0\}$.

Example 4.4.6. If σ is an honest time and the assumptions (AC) are satisfied, then the Azéma supermartingale of σ decomposes as

$$Z_t^{\mathsf{P}} = \mathsf{P}(\sigma > t|\mathcal{F}_t) = \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}, \quad \text{i.e. } D_t^{\mathsf{P}} = \frac{1}{\overline{N}_t^{\mathsf{P}}},$$

where N^{P} is a non-negative local martingale converging to zero almost surely, cf. Lemma 4.3.9.

Theorem 4.4.7. *Assume that $\rho > 0$ almost surely and that $\widetilde{\rho}_t := \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{G}_t)$ is continuous. If $Z^{\mathsf{P}} = N^{\mathsf{P}}D^{\mathsf{P}}$ and $Z^{\mathsf{Q}} = N^{\mathsf{Q}}D^{\mathsf{Q}}$ denote the Itô-Watanabe decompositions of the Azéma supermartingales of σ under P and Q, then $D_t^{\mathsf{P}} = D_t^{\mathsf{Q}}$ for all $t \geq 0$ almost surely on the set $\{Z^{\mathsf{P}} > 0\} = \{Z^{\mathsf{Q}} > 0\}$.*

Proof. Corollary 4.4.3 implies the existence of a local $(\mathsf{P}, \mathcal{F}_t)$-martingale M such that

$$\begin{aligned} h_t &= \mathbb{E}^{\mathsf{P}}\left(\widetilde{\rho}_t \mathbb{1}_{\{\sigma > t\}}\middle|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\left(M_t - \int_0^t \frac{d\langle N^{\mathsf{P}}, M\rangle_s}{N_s^{\mathsf{P}}}\right)\mathbb{1}_{\{\sigma > t\}}\middle|\mathcal{F}_t\right) \\ &= \left(M_t - \int_0^t \frac{d\langle N^{\mathsf{P}}, M\rangle_s}{N_s^{\mathsf{P}}}\right) Z_t^{\mathsf{P}}. \end{aligned}$$

Hence,

$$\rho_t N_t^{\mathsf{Q}} D_t^{\mathsf{Q}} = \rho_t Z_t^{\mathsf{Q}} = h_t = \left(M_t - \int_0^t \frac{d\langle N^{\mathsf{P}}, M\rangle_s}{N_s^{\mathsf{P}}}\right) N_t^{\mathsf{P}} D_t^{\mathsf{P}}.$$

Obviously, we have $\{Z^{\mathsf{P}} > 0\} = \{h > 0\} = \{Z^{\mathsf{Q}} > 0\}$. Moreover, the process

$$\left(M_t - \int_0^t \frac{d\langle N^{\mathsf{P}}, M\rangle_s}{N_s^{\mathsf{P}}}\right) N_t^{\mathsf{P}} = M_0 N_0^{\mathsf{P}} + \int_0^t N_s^{\mathsf{P}} dM_s + \int_0^t \left(M_s - \int_0^s \frac{d\langle N^{\mathsf{P}}, M\rangle_u}{N_u^{\mathsf{P}}}\right) dN_s^{\mathsf{P}}$$

is a non-negative local $(\mathsf{P}, \mathcal{F}_t)$-martingale. Since $(\rho_t N_t^{\mathsf{Q}})$ is also a non-negative local $(\mathsf{P}, \mathcal{F}_t)$-martingale, the uniqueness of the Itô-Watanabe decomposition yields that

$$\left(M_t - \int_0^t \frac{d\langle N^{\mathsf{P}}, M\rangle_s}{N_s^{\mathsf{P}}}\right) N_t^{\mathsf{P}} = \rho_t N_t^{\mathsf{Q}} \quad \text{and} \quad D_t^{\mathsf{P}} = D_t^{\mathsf{Q}}$$

for all $t \geq 0$ almost surely on $\{Z^{\mathsf{P}} > 0\} = \{Z^{\mathsf{Q}} > 0\}$. $\square$

Remark 4.4.8. Note that if $Z_t > 0$ a.s. for all $t \geq 0$ one can write $Z_t = N_t \cdot \exp(-\Lambda_t)$, where $\Lambda_t := -\ln(D_t)$ is referred to as the intensity process in the credit risk literature. Intuitively, to affect the intensity of σ via a change of measure the Radon-Nikodym density should involve a stochastic integral with respect to the discontinuous martingale $\mathbb{1}_{\{\sigma \leq t\}} - A_t$. Indeed, the above theorem shows that a change of measure via a continuous $(\mathcal{G}_t)$-martingale will not change the intensity process. See also Theorem 6.4 in [13].

Indeed, the following counterexample shows that the assumption that $(\widetilde{\rho}_t)$ is continuous cannot be dropped in Theorem 4.4.7.

Counterexample 4.4.9. Let σ be an honest time. We then have according to Lemma 4.3.9

$$\sigma = \sup\{t > 0 : \; N_t^{\mathsf{P}} = \overline{N}_t^{\mathsf{P}}\}, \qquad Z_t^{\mathsf{P}} = \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}$$

for some non-negative local martingale N^{P} converging to zero. Take $\rho = \log(\overline{N}_\infty^{\mathsf{P}}) = \log(N_\sigma^{\mathsf{P}}) = A_\sigma^{\mathsf{P}}$.

$$\begin{aligned}
\widetilde{\rho}_t = \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{G}_t) &= \mathbb{1}_{\{\sigma\le t\}}\log(\overline{N}_t^{\mathsf{P}}) + \mathbb{1}_{\{\sigma>t\}}\frac{\mathbb{E}^{\mathsf{P}}\left(\log(\overline{N}_\infty^{\mathsf{P}})\mathbb{1}_{\{\sigma>t\}}\middle|\mathcal{F}_t\right)}{Z_t^{\mathsf{P}}} \\
&= \mathbb{1}_{\{\sigma\le t\}}\log(\overline{N}_t^{\mathsf{P}}) + \mathbb{1}_{\{\sigma>t\}}\frac{\overline{N}_t^{\mathsf{P}}}{N_t^{\mathsf{P}}}\cdot h_t \\
&= \mathbb{1}_{\{\sigma\le t\}}\log(\overline{N}_t^{\mathsf{P}}) + \mathbb{1}_{\{\sigma>t\}}\frac{\overline{N}_t^{\mathsf{P}}}{N_t^{\mathsf{P}}}\cdot N_t^{\mathsf{P}}\int_{\log(\overline{N}_t^{\mathsf{P}})}^{\infty} xe^{-x}dx \\
&= \mathbb{1}_{\{\sigma\le t\}}\log(\overline{N}_t^{\mathsf{P}}) + \mathbb{1}_{\{\sigma>t\}}(1+\log(\overline{N}_t^{\mathsf{P}})) = \log(\overline{N}_t^{\mathsf{P}}) + \mathbb{1}_{\{\sigma>t\}},
\end{aligned}$$

where h_t has already been computed in Example 4.3.11. Hence, $\widetilde{\rho}$ is a purely discontinuous $(\mathsf{P},\mathcal{G}_t)$-martingale and

$$\rho_t = \log(\overline{N}_t^{\mathsf{P}}) + \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}.$$

Therefore,

$$Z_t^{\mathsf{Q}} = \frac{h_t}{\rho_t} = \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}(1+\log(\overline{N}_t^{\mathsf{P}}))\frac{1}{\rho_t} = \frac{N_t^{\mathsf{P}}}{\overline{N}_t^{\mathsf{P}}}(1+\log(\overline{N}_t^{\mathsf{P}}))\frac{\overline{N}_t^{\mathsf{P}}}{N_t^{\mathsf{P}} + \overline{N}_t^{\mathsf{P}}\log(\overline{N}_t^{\mathsf{P}})} = \frac{N_t^{\mathsf{P}} + N_t^{\mathsf{P}}\log(\overline{N}_t^{\mathsf{P}})}{N_t^{\mathsf{P}} + \overline{N}_t^{\mathsf{P}}\log(\overline{N}_t^{\mathsf{P}})}.$$

And since $N^{\mathsf{P}}/\rho \in \mathcal{M}_{loc}(\mathsf{Q},\mathcal{F}_t)$, the Itô-Watanabe decomposition of Z^{Q} takes the form

$$Z_t^{\mathsf{Q}} = \frac{N_t^{\mathsf{P}}}{\rho_t}\cdot\frac{1+\log(\overline{N}_t^{\mathsf{P}})}{\overline{N}_t^{\mathsf{P}}} = N_t^{\mathsf{Q}}D_t^{\mathsf{Q}}$$

with

$$D_t^{\mathsf{Q}} = \frac{1+\log(\overline{N}_t^{\mathsf{P}})}{\overline{N}_t^{\mathsf{P}}} \neq \frac{1}{\overline{N}_t^{\mathsf{P}}} = D_t^{\mathsf{P}} \quad \forall\, t>0.$$

Remark 4.4.10. In view of Examples 4.3.4 and 4.3.11 one may wonder whether for an arbitrary random time σ the measure change $\rho = f(A_\sigma^{\mathsf{P}})$ implies that $\mu^L = N^{\mathsf{P}}$ P-almost surely, where $f(\cdot) > 0$ is chosen such that $\mathbb{E}^{\mathsf{P}}\rho = 1$. In all generality, one can compute

$$\begin{aligned}
h_t &= \mathbb{E}^{\mathsf{P}}\left(f(A_\sigma^{\mathsf{P}})\mathbb{1}_{\{\sigma>t\}}\middle|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(\int_t^\infty f(A_u^{\mathsf{P}})dA_u^{\mathsf{P}}\middle|\mathcal{F}_t\right) \\
&= \int_{A_t^{\mathsf{P}}}^\infty \mathsf{P}(A_\infty^{\mathsf{P}}\in dx|\mathcal{F}_t)\int_{A_t^{\mathsf{P}}}^x f(y)dy = \int_{A_t^{\mathsf{P}}}^\infty f(y)\int_y^\infty \mathsf{P}(A_\infty^{\mathsf{P}}\in dx|\mathcal{F}_t)dy \\
&= \int_{A_t^{\mathsf{P}}}^\infty f(y)\mathsf{P}(A_\infty^{\mathsf{P}} > y|\mathcal{F}_t)dy = \int_0^\infty f(y+A_t^{\mathsf{P}})\cdot\mathsf{P}(A_\infty^{\mathsf{P}} - A_t^{\mathsf{P}} > y|\mathcal{F}_t)dy.
\end{aligned}$$

Now let us write for all $t, y \ge 0$,

$$\mathsf{P}(A_\infty^{\mathsf{P}} - A_t^{\mathsf{P}} > y|\mathcal{F}_t) = N_t^{\mathsf{P}}\cdot F_t(y)$$

for some measurable function $F_t(y)(\omega) : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \geq 0$,

$$D_t^{\mathsf{P}} = \int_0^\infty F_t(y)dy.$$

Then in order to have $\mu^L = N^{\mathsf{P}}$ P-a.s. it is sufficient to require that $F_t(y)$ is of the form $g(y, L_t)$ for some deterministic measurable function g, where L is a d-dimensional semimartingale of finite variation.

4.5 Invariance of pseudo-stopping times

Since the definition of a pseudo-stopping time depends on the underlying probability measure, one may wonder whether there exist equivalent changes of probability measure which preserve the pseudo-stopping time property. Let us look at an example.

Example 4.5.1. For a standard $(\mathsf{P}, \mathcal{F}_t)$-Brownian motion B define for all $a \in \mathbb{R}$ and $s \geq 0$ the stopping time

$$\tau_s^a := \inf\{t > s : \ B_t = a\}$$

as well as

$$L := \sup\{t < \tau_0^1 : \ B_t = 0\}, \qquad \sigma := \sup\{t < L : \ \overline{B}_t = B_t\} = \sup\{t < L : \ \overline{B}_L = B_t\}.$$

It is well-known that σ is a $(\mathsf{P}, \mathcal{F}_t)$-pseudo-stopping time. Let $b : \mathbb{R} \to \mathbb{R}$ be a bounded function and set

$$\rho_t = \mathcal{E}\left(\int_0^t b(B_s)dB_s\right).$$

Then $(\rho_t)_{t\geq 0}$ is a positive $(\mathsf{P}, \mathcal{F}_t)$-martingale which under some technical conditions on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ defines a measure Q on $\mathcal{F}_\infty$ such that

$$\left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_{\mathcal{F}_t} = \rho_t, \quad \forall\, t \geq 0.$$

Note that in general Q is only locally equivalent to P, i.e. it may be singular to P on $\mathcal{F}_\infty$. By Girsanov's theorem the process

$$W_t := B_t - \int_0^t b(B_s)ds$$

is a Q-Brownian motion and B is an Itô-diffusion. We denote its Q-scale function by $s(\cdot)$. Using the Markov property of B we compute the Q-Azéma supermartingale of L as

$$Z_t^{L,\mathsf{Q}} := \mathsf{Q}(L > t|\mathcal{F}_t) = \mathsf{Q}\left(\tau_t^1 > \tau_t^0|\mathcal{F}_t\right) = \frac{s(1) - s(B_t)}{s(1) - s(0)} \wedge 1.$$

Since s is an increasing function,

$$\sigma = \sup\{t < L : \ \overline{B}_L = B_t\} = \sup\left\{t < L : \ s(\overline{B}_L) = s(B_t)\right\} = \sup\left\{t < L : \ \underline{Z}_L^{L,\mathsf{Q}} = Z_t^{L,\mathsf{Q}}\right\}.$$

According to Proposition 5 in [58], σ is thus also a Q-pseudo-stopping time and

$$Z_t^{\sigma,\mathsf{Q}} := \mathsf{Q}(\sigma > t|\mathcal{F}_t) = \underline{Z}_t^{L,\mathsf{Q}}.$$

The previous example suggests that pseudo-stopping times can be robust with respect to (locally) equivalent changes of probability. Another class of examples can be found in a credit risk setting via the so called Cox construction.

Example 4.5.2. Assume that there exists a random variable U which is independent of $\mathcal{F}_\infty$ such that $\mathsf{P}(U > t) = \exp(-t)$ for all $t \geq 0$. Let (Λ_t) be an $(\mathcal{F}_t)$-adapted continuous increasing process and define

$$\sigma := \inf\{t \geq 0 : \ \Lambda_t \geq U\}.$$

Then,

$$Z_t^{\mathsf{P}} = \mathsf{P}(\sigma > t|\mathcal{F}_t) = \mathsf{P}(\Lambda_t < U|\mathcal{F}_t) = \exp(-\Lambda_t).$$

Let $\rho \in \mathcal{F}_\infty$ be a strictly positive random variable with $\mathbb{E}^{\mathsf{P}}\rho = 1$ defining the equivalent measure $\mathsf{Q} := \rho.\mathsf{P}$. Then also

$$\begin{aligned} Z_t^{\mathsf{Q}} &= \frac{h_t}{\rho_t} = \frac{\mathbb{E}^{\mathsf{P}}(\mathbb{E}^{\mathsf{P}}(\rho \mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_\infty)|\mathcal{F}_t)}{\rho_t} = \frac{\mathbb{E}^{\mathsf{P}}(\rho \cdot \mathsf{P}(\Lambda_t < U|\mathcal{F}_\infty)|\mathcal{F}_t)}{\rho_t} \\ &= \frac{\mathbb{E}^{\mathsf{P}}(\rho \exp(-\Lambda_t)|\mathcal{F}_t)}{\rho_t} = \exp(-\Lambda_t). \end{aligned}$$

Hence, σ is a P- and Q-pseudo-stopping time. Moreover, $Z^{\mathsf{Q}} = Z^{\mathsf{P}}$ almost surely.

However, in general the pseudo-stopping time property is not robust with respect to general measure changes as the following counterexample shows.

Counterexample 4.5.3. Let σ be an $\mathcal{F}_\infty$-measurable $(\mathsf{P}, \mathcal{F}_t)$-pseudo-stopping time and define the random variable $\rho = 2Z_\sigma^{\mathsf{P}} \in \mathcal{F}_\infty$. Since $Z_\sigma^{\mathsf{P}} \sim \mathcal{U}[0,1]$, the measure $\mathsf{Q} = \rho.\mathsf{P}$ is well-defined and equivalent to P.

$$\begin{aligned} \widetilde{\rho}_t &= 2\mathbb{E}^{\mathsf{P}}(Z_\sigma^{\mathsf{P}}|\mathcal{G}_t) = 2\mathbb{1}_{\{\sigma \leq t\}} Z_\sigma^{\mathsf{P}} + 2\mathbb{1}_{\{\sigma>t\}} \frac{\mathbb{E}^{\mathsf{P}}(Z_\sigma^{\mathsf{P}} \mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t)}{Z_t^{\mathsf{P}}} \\ &= 2\mathbb{1}_{\{\sigma \leq t\}} Z_\sigma^{\mathsf{P}} + 2\mathbb{1}_{\{\sigma>t\}} \frac{\mathbb{E}^{\mathsf{P}}\left(\int_t^\infty (1-A_u^{\mathsf{P}}) dA_u^{\mathsf{P}} \middle| \mathcal{F}_t\right)}{Z_t^{\mathsf{P}}} \\ &= 2\mathbb{1}_{\{\sigma \leq t\}} Z_\sigma^{\mathsf{P}} + \mathbb{1}_{\{\sigma>t\}} \frac{(1-A_t^{\mathsf{P}})^2}{Z_t^{\mathsf{P}}} = Z_{t\wedge\sigma}^{\mathsf{P}} + \mathbb{1}_{\{\sigma \leq t\}} Z_\sigma^{\mathsf{P}}, \end{aligned}$$

which jumps at time σ. Moreover,

$$h_t = \mathbb{E}^{\mathsf{P}}(\widetilde{\rho}_t \mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t) = Z_t^{\mathsf{P}} \cdot \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t) = (Z_t^{\mathsf{P}})^2.$$

Since $\rho = \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{F}_\infty) = \rho_\infty \neq 1$ almost surely, the continuous uniformly integrable martingale (ρ_t) is not identical to one. Therefore, having in mind that (Z_t^{P}) is of finite variation,

$$Z_t^{\mathsf{Q}} = \frac{h_t}{\rho_t} = \frac{(Z_t^{\mathsf{P}})^2}{\rho_t}$$

cannot be of finite variation, which implies that σ is *not* a Q-pseudo-stopping time.

On the other hand, suppose that there exists a measure Q such that σ is a P- and Q-pseudo-stopping time. In this case, if $\widetilde{\rho}_t$ is strictly positive and continuous, Theorem 4.4.7 would imply that $Z_t^{\mathsf{Q}} = Z_t^{\mathsf{P}}$ almost surely for all $t \geq 0$. This observation led us to look for possible measure changes ρ which preserve the pseudo-stopping time property for the class of P-pseudo-stopping times we dealt with in Subsection 4.3.3.

Theorem 4.5.4. *Let L be an honest time. Then $Z_t^L := \mathsf{P}(L > t|\mathcal{F}_t) = M_t/\overline{M}_t$ for some non-negative local P-martingale M with $M_0 = 1$, converging to zero almost surely. Define*

$$\sigma := \sup\left\{t < L : \ Z_t^L = \inf_{u \leq L} Z_u^L\right\}.$$

Moreover, let $g : [0,1] \to \mathbb{R}$ be a Lebesgue-integrable function which satisfies

$$\int_0^1 \exp\left(\int_z^1 g(y)dy\right) dz = 1.$$

If the process

$$\rho_t := \mathcal{E}\left(\int_0^t g\left(\frac{M_s}{\overline{M}_s}\right)\frac{dM_s}{2\overline{M}_s}\right)$$

is a uniformly integrable $(\mathsf{P}, \mathcal{F}_t)$-martingale, then there exists a measure $\mathsf{Q} \sim \mathsf{P}$ such that σ is a pseudo-stopping time with respect to P and Q. Moreover, in this case the identity $Z_t^{\mathsf{P}} = \mathsf{P}(\sigma > t|\mathcal{F}_t) = \mathsf{Q}(\sigma > t|\mathcal{F}_t) = Z_t^{\mathsf{Q}}$ is satisfied almost surely.

Proof. From [58], Proposition 5, it is known that σ is a P-pseudo-stopping time. We set $\mathsf{Q} = \rho_\infty.\mathsf{P}$ and define

$$N_t := h\left(\frac{M_t}{\overline{M}_t}\right)\cdot \overline{M}_t,$$

where $h : [0,1] \to \mathbb{R}_+$ is the function

$$h(x) = \int_0^x \exp\left(\int_z^1 g(y)dy\right) dz.$$

Note that h satisfies

$$g(x)h'(x) + h''(x) = 0, \quad h(1) = h'(1) = 1, \quad h(0) = 0.$$

This implies that N is a local $(\mathsf{Q}, \mathcal{F}_t)$-martingale. Indeed, by Girsanov's theorem

$$\widetilde{M_t} := M_t - \int_0^t g\left(\frac{M_s}{\overline{M}_s}\right)\frac{d\langle M\rangle_s}{2\overline{M}_s}$$

is a local Q-martingale and

$$\begin{aligned} dN_t &= h\left(\frac{M_t}{\overline{M}_t}\right)d\overline{M}_t + \overline{M}_t h'\left(\frac{M_t}{\overline{M}_t}\right)\left[\frac{dM_t}{\overline{M}_t} - \frac{d\overline{M}_t}{\overline{M}_t}\right] + \frac{1}{2}h''\left(\frac{M_t}{\overline{M}_t}\right)\frac{d\langle M\rangle_t}{\overline{M}_t} \\ &= [h(1) - h'(1)]d\overline{M}_t + h'\left(\frac{M_t}{\overline{M}_t}\right)d\widetilde{M}_t. \end{aligned}$$

Furthermore, h is strictly increasing. Therefore, $\overline{M} = \overline{N}$ and

$$L = \sup\{t > 0 : \; M_t = \overline{M}_t\} = \sup\{t > 0 : \; N_t = \overline{N}_t\}.$$

Since $N_t \to 0$ almost surely,

$$Z_t^{\mathsf{Q},L} := \mathsf{Q}(L > t|\mathcal{F}_t) = \frac{N_t}{\overline{N}_t} = h\left(\frac{M_t}{\overline{M}_t}\right)\frac{\overline{M}_t}{\overline{N}_t} = h\left(\frac{M_t}{\overline{M}_t}\right) = h(Z_t^L).$$

But then

$$\sigma = \sup\left\{t < L : \; Z_t^L = \inf_{u\le L} Z_u^L\right\} = \sup\left\{t < L : \; Z_t^{\mathsf{Q},L} = \inf_{u \le L} Z_u^{\mathsf{Q},L}\right\}$$

and σ is a Q-pseudo stopping time by Proposition 5 of [58]. □

Even if (ρ_t) is not a uniformly integrable martingale in the setting of the previous theorem, there exists a measure Q (under some technical conditions on the probability space) such that σ is not only a P- but also a Q-pseudo-stopping time. However, the measure Q will only be a dominating measure in general which is known as the so called Föllmer measure associated with (ρ_t). Nevertheless, it is also not hard to find an example which satisfies the integrability assumption as one can see below.

Example 4.5.5. Consider $g(x) = x - c$, where $c > 0$ is chosen such that

$$\int_0^1 \exp\left(\frac{1-z^2}{2} - c(1-z)\right) dz = 1.$$

Using product integration,

$$\begin{aligned} \int_0^t \frac{M_t}{\overline{M}_t^2} dM_t &= \frac{M_t^2}{\overline{M}_t^2} - 1 - \int_0^t M_t \left(\frac{dM_t}{\overline{M}_t^2} - \frac{2M_t}{\overline{M}_t^3} d\overline{M}_t\right) - \int_0^t \frac{d\langle M\rangle_t}{\overline{M}_t^2} \\ &\le -\int_0^t \frac{M_t}{\overline{M}_t^2} dM_t + 2\int_0^t \frac{d\overline{M}_t}{\overline{M}_t} \\ \Leftrightarrow \quad \int_0^t \frac{M_t}{\overline{M}_t^2} dM_t &\le \log(\overline{M}_t), \\ X_t &:= \int_0^t \frac{dM_t}{\overline{M}_t} = \frac{M_t}{\overline{M}_t} - 1 + \int_0^t \frac{M_t}{\overline{M}_t^2} d\overline{M}_t = \frac{M_t}{\overline{M}_t} - 1 + \log(\overline{M}_t) \ge -1, \\ Y_t &:= \int_0^t g\left(\frac{M_t}{\overline{M}_t}\right) \frac{dM_t}{\overline{M}_t} = \int_0^t \frac{M_t}{\overline{M}_t^2} dM_t - c\int_0^t \frac{dM_t}{\overline{M}_t} \le c + \log(\overline{M}_t) \\ &\le c + \log(\overline{M}_\infty). \end{aligned}$$

First note that $X = (X_t)$ is a uniformly integrable martingale bounded from below, since

$$\mathbb{E}^{\mathsf{P}} X_\infty = 0 - 1 + \mathbb{E}^{\mathsf{P}} \log(\overline{M}_\infty) = 0 - 1 + 1 = 0,$$

where we have used the fact that $\log(\overline{M}_\infty) \sim \mathrm{Exp}(1)$, cf. Lemma 4.3.10. Moreover,

$$\sup_{t\ge 0} \mathbb{E}^{\mathsf{P}} X_t^2 \le \mathbb{E}^{\mathsf{P}} \left(1 + \log(\overline{M}_\infty)\right)^2 = \int_0^\infty (1+x)^2 e^{-x} dx = 5.$$

Therefore X is square-integrable and

$$\mathbb{E}^{\mathsf{P}} \langle Y\rangle_\infty = \mathbb{E}^{\mathsf{P}} \int_0^\infty g^2\left(\frac{M_t}{\overline{M}_t}\right) d\langle X\rangle_t \le (1+c)^2 \cdot \mathbb{E}^{\mathsf{P}} \langle X\rangle_\infty < \infty.$$

By the Burkholder-Davis-Gundy inequality thus $\mathbb{E}^{\mathsf{P}} \sup_{t\ge 0} |Y_t| < \infty$ and the dominated convergence theorem yields the martingality of $Y = (Y_t)$. Moreover for all $t \ge 0$,

$$\mathbb{E}^{\mathsf{P}} \exp\left(\frac{Y_t}{2}\right) \le e^{c/2} \cdot \mathbb{E}^{\mathsf{P}} \exp\left(\frac{\log(\overline{M}_\infty)}{2}\right) = e^{c/2} \cdot \mathbb{E}^{\mathsf{P}} \sqrt{\overline{M}_\infty} = e^{c/2} \cdot \int_0^1 \frac{dx}{\sqrt{x}} = 2e^{c/2}.$$

Hence, by Jensen's inequality $(\exp(Y_t/2))_{t\ge 0}$ is a uniformly integrable submartingale and Kazamaki's criterion implies the uniform integrability of (ρ_t).

4.6 Financial applications: no arbitrage up to a random time?

In the following we will work with a financial market model consisting of one risky security S and a risk free bond. For simplicity, we assume that the interest rate is equal to zero. We suppose that the market satisfies NFLVR and w.l.o.g. P is assumed to be the risk-neutral measure, i.e. S is a positive local $(\mathsf{P}, \mathcal{F}_t)$-martingale. A natural question is now, if the market is still arbitrage free after adding new information by enlarging the filtration progressively with σ.

In the case where σ is an honest time this question has been discussed in details by [31]. Furthermore, it is known that NFLVR fails if S does not remain a semimartingale in the enlarged filtration. Since this is only clear until time σ, we will in the following restrict ourselves to the question whether the market $(S_{t\wedge\sigma}, \mathcal{G}_{t\wedge\sigma}, \mathsf{P})$ is arbitrage-free. Note also that the question of the existence of an equivalent local martingale measure on the whole time horizon $[0,\infty)$ has previously been addressed in [13], where its connection to the so called $(\mathcal{H})$-hypothesis has been pointed out.

For the reader's convenience we first repeat some notions commonly used in finance: An a-admissible trading strategy for the $(\mathcal{F}_t)$-adapted price process (S_t) is any $(\mathcal{F}_t)$-predictable process (θ_t), which is (S_t) integrable such that the value process

$$V(x,\theta)_t := x + \int_0^t \theta_s dS_s$$

satisfies $V(0,\theta)_t \geq -a$ P-almost surely for all $t \geq 0$. A trading strategy is admissible if it is a-admissible for some $a \in \mathbb{R}_+$. The notion of admissibility allows us to define two different no arbitrage concepts.

Definition 4.6.1. In the market model $(S_t, \mathcal{F}_t, \mathsf{P})$ there is

- an Arbitrage of the First Kind on $[0,T]$ for $T \in (0,\infty)$ if and only if there exists a non-negative $\mathcal{F}_T$-measurable random variable ξ with $\mathsf{P}(\xi > 0) > 0$ such that for all $a > 0$ there exists an a-admissible trading strategy θ such that $V(a,\theta)_T \geq \xi$ almost surely. If there is no arbitrage of the first kind on any interval $[0,T]$, $T \in (0,\infty)$, we say that the market satisfies the NA1 (No Arbitrage of the First Kind) condition.
- a Free Lunch with Vanishing Risk (FLVR) if and only if there exists an $\varepsilon > 0$ and a sequence (θ^n) of $(\mathcal{F}_t)$-admissible strategies together with an increasing sequence (δ_n) of positive numbers converging to one such that $\mathsf{P}(V(0,\theta^n)_\infty > -1 + \delta_n) = 1$ and $\mathsf{P}(V(0,\theta^n)_\infty > \varepsilon) \geq \varepsilon$. Otherwise we say that the market satisfies the NFLVR (No Free Lunch with Vanishing Risk) condition.

Theorem 4.6.4 below gives the connection between the above no arbitrage criteria with the dual variables defined in

Definition 4.6.2. In the market model $(S_t, \mathcal{F}_t, \mathsf{P})$ we call

- a strictly positive local $(\mathcal{F}_t)$-martingale (L_t) with $L_0 = 1$ and $L_\infty > 0$ almost surely a local martingale deflator, if the process $(L_t S_t)$ is a local $(\mathcal{F}_t)$-martingale.
- $\widetilde{\mathsf{P}} := L_\infty.\mathsf{P}$ an Equivalent Local Martingale Measure (ELMM), if there exists a local martingale deflator (L_t) which is a uniformly integrable martingale closed by L_∞.

Remark 4.6.3. NA1 is also known under the acronym NUPBR (No Unbounded Profit with Bounded Risk).

The following theorem contains results which are non-trivial but well-known. Their proofs can be found in [19] and [69], cf. also Proposition 2.3 in [2].

Theorem 4.6.4. *In the financial market model* $(S_t, \mathcal{F}_t, \mathsf{P})$

- *the NA1 condition is equivalent to the existence of a local martingale deflator.*
- *the NFLVR condition is equivalent to the existence of an ELMM.*

For the enlarged market model $(S_{t\wedge\sigma}, \mathcal{G}_{t\wedge\sigma}, \mathsf{P})$ things can be defined in an analogous way. For notational convenience we will write $Z = m - A$ instead of $Z^{\mathsf{P}} = m^{\mathsf{P}} - A^{\mathsf{P}}$ for the Azéma supermartingale of σ under P for the rest of this section.

The following theorem gives a necessary criterion to have NFLVR on the time horizon $[0, T \wedge \sigma]$, where T is a $(\mathcal{G}_t)$-stopping time. In the case of σ being an honest time the following statement can be found in [31] together with a long technical proof. However, we will give an apparently new proof of the statement, valid for all random times, which appeals to purely intuitive reasoning.

Theorem 4.6.5. *Let T be a $(\mathcal{G}_t)$-stopping time. If $\mathsf{P}(T_0^Z \le T) = 0$, then NFLVR also holds in the enlarged financial market on the time horizon* $[0, \sigma \wedge T]$.

The idea of the proof is that even at time T we cannot be sure that σ has already occured. In fact σ may still happen only after the stopping time T because $\mathsf{P}(T_0^Z \le T) = 0$.

Proof. W.l.o.g. we may assume that T is actually an $(\mathcal{F}_t)$-stopping time, cf. Lemma 4.3.1. Note that the condition $\mathsf{P}(T_0^Z \le T) = 0$ is in fact equivalent to

$$\mathsf{P}(\forall\, t \in [0, T] :\ Z_t > 0) = 1.$$

We proceed by contradiction: Assume that there is a FLVR in the enlarged market on the time horizon $[0, \sigma \wedge T]$. Then there exists a sequence of $(\mathcal{G}_t)$-admissible trading strategies $(\theta^n)_{n\in\mathbb{N}}$ and an increasing deterministic sequence (δ_n) converging towards 1 such that for some $\varepsilon > 0$ and all $n \in \mathbb{N}$,

$$\mathsf{P}\left(V(0, \theta^n)_{\sigma\wedge T} > -1 + \delta_n\right) = 1, \quad \mathsf{P}\left(V(0, \theta^n)_{\sigma\wedge T} > \varepsilon\right) \ge \varepsilon.$$

With the help of Lemma 4.3.1 we can find for every $n \in \mathbb{N}$ an $(\mathcal{F}_t)$-predictable process (y_t^n) such that

$$\theta_t^n \mathbb{1}_{\{t\le\sigma\}} = y_t^n \mathbb{1}_{\{t\le\sigma\}}.$$

We will prove that

$$\mathsf{P}(V(0, y^n)_T > -1 + \delta_n) = 1.$$

Assume that this was not the case, i.e.

$$\mathsf{P}(V(0, y^n)_T \le -1 + \delta_n) > 0.$$

Since $Z_T > 0$ almost surely, this would imply that

$$\begin{aligned} 0 < \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_{\{V(0,y^n)_T \le -1+\delta_n\}} Z_T\right) &= \mathsf{P}(V(0, y^n)_T \le -1 + \delta_n;\ \sigma > T) \\ &= \mathsf{P}(V(0, \theta^n)_{\sigma\wedge T} \le -1 + \delta_n;\ \sigma > T) \\ &\le \mathsf{P}(V(0, \theta^n)_{\sigma\wedge T} \le -1 + \delta_n) = 0, \end{aligned}$$

a contradiction. Similarly one shows that for all $t \geq 0$,

$$\mathsf{P}(V(0,y^n)_{t\wedge T} > -a_n) = 1$$

for some $a_n \in \mathbb{R}_+$, because each θ^n is assumed to be admissible. Thus, each y^n is admissible as well. For every $n \in \mathbb{N}$ we define the $(\mathcal{F}_t)$-trading strategy

$$\vartheta^n_t := y^n_t \mathbb{1}_{\{0\leq t\leq T^n_\varepsilon\}},$$

where

$$T^n_\varepsilon := \inf\{t \geq 0 : \ V(0,y^n)_t = \varepsilon\}.$$

Obviously, ϑ^n is admissible and

$$\mathsf{P}(V(0,\vartheta^n)_T > -1+\delta_n) \geq \mathsf{P}(V(0,y^n)_T > -1+\delta_n) = 1.$$

Moreover,

$$\begin{aligned}
\mathsf{P}\left(V(0,\vartheta^n)_T > \frac{\varepsilon}{2}\right) &\geq \mathsf{P}\left(T^n_\varepsilon \leq T\right) = \mathsf{P}\left(\exists\, u \leq T : \ V(0,y^n)_u \geq \varepsilon\right) \\
&\geq \mathsf{P}\left(\exists\, u \leq \sigma\wedge T : \ V(0,y^n)_u \geq \varepsilon\right) \\
&= \mathsf{P}\left(\exists\, u \leq \sigma\wedge T : \ V(0,\theta^n)_u \geq \varepsilon\right) \\
&\geq \mathsf{P}\left(V(0,\theta^n)_{\sigma\wedge T} \geq \varepsilon\right) > \varepsilon.
\end{aligned}$$

Choosing $\widetilde{\varepsilon} := \varepsilon/2$, this would give a FLVR with respect to $(\mathcal{F}_t)$ and thus a contradiction because S is assumed to be a local $(\mathsf{P},\mathcal{F}_t)$-martingale. □

In [31] it is moreover shown that the condition $\mathsf{P}(T^Z_0 \leq T) = 0$ is not only sufficient but also necessary to have NFLVR on $[0, T\wedge\sigma]$, if σ is honest and the market is complete. However, the condition $\mathsf{P}(T^Z_0 \leq T) = 0$ is not in general necessary, even in a complete market, as the following example shows.

Example 4.6.6. Let σ be a pseudo-stopping time bounded by one. Then $1 - Z_1 = A_1 = 1$ and therefore $\mathsf{P}(T^Z_0 \leq 1) = 1$. However, since σ is a pseudo-stopping time $(S_{t\wedge\sigma})$ is a local $(\mathcal{G}_t)$-martingale and therefore NFLVR holds in the enlarged market on the interval $[0,\sigma] = [0,\sigma\wedge 1]$.

The following Lemma was proven in [31] in the case of honest times, where it was remarked that it also holds in more generality. For completeness we provide a proof as well.

Lemma 4.6.7. *The process* $(1/N_{t\wedge\sigma})_{t\geq 0}$ *is a local martingale deflator for* $(S_{t\wedge\sigma})$ *in the filtration* $(\mathcal{G}_t)$*, i.e.* $NA1$ *holds with respect to* $(\mathcal{G}_t)$ *on the time horizon* $[0,\sigma]$.

Proof. First note that the process $1/N_{t\wedge\sigma}$ is well-defined, since $Z_\sigma = Z_{\sigma-} > 0$, cf. [42]. From the enlargement formula the processes

$$\widetilde{S}_t = S_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{d\langle S,N\rangle_s}{N_s}$$

and

$$\widetilde{N}_t = N_{t\wedge\sigma} - \int_0^{t\wedge\sigma} \frac{d\langle N\rangle_s}{N_s}$$

are local $(\mathsf{P},\mathcal{G}_t)$-martingales. With Itô's formula we then have on $[0,\sigma]$,

$$\begin{aligned}
d\left(\frac{S}{N}\right) &= \frac{dS}{N} - \frac{S}{N^2}dN + \frac{S}{N^3}d\langle N\rangle - \frac{d\langle S,N\rangle}{N^2} \\
&= \frac{S}{N}\left(\frac{d\widetilde{S}}{S} + \frac{d\langle S,N\rangle}{SN} - \frac{d\widetilde{N}}{N} - \frac{d\langle N\rangle}{N^2} + \frac{d\langle N\rangle}{N^2} - \frac{d\langle S,N\rangle}{SN}\right) = \frac{S}{N}\left(\frac{d\widetilde{S}}{S} - \frac{d\widetilde{N}}{N}\right).
\end{aligned}$$

Especially, taking $S \equiv 1$ yields that $1/N_{t\wedge\sigma} \in \mathcal{M}_{loc}(\mathsf{P},\mathcal{G}_t)$. □

Remark 4.6.8. The validity of NA1 in a progressively enlarged filtration has recently been proven to hold in much greater generality without assuming (AC), cf. [1, 2, 3].

Next we prove a sufficient and necessary criterion such that $1/N_{t\wedge\sigma}$ is a uniformly integrable martingale on the time interval $[0,\sigma\wedge T]$, where T is a $(\mathcal{G}_t)$-stopping time. For this we need the Itô-Watanabe decomposition of the Azéma supermartingale of σ which we denote as before by $Z = ND$, where N and D are defined as in Remark 4.4.5.

Theorem 4.6.9. *Let T be a $(\mathcal{G}_t)$-stopping time. Then,*

$$\left(\frac{1}{N_{t\wedge\sigma\wedge T}}\right) \in \mathcal{M}_{u.i.}(\mathsf{P},\mathcal{G}_t) \quad \Leftrightarrow \quad \mathsf{P}\left(T \geq T_0^N < T_0^D\right) = 0.$$

Proof. The local $(\mathcal{G}_t)$-martingale $(1/N_{t\wedge\sigma\wedge T})_{t\geq 0}$ is a uniformly integrable martingale if and only if $\mathbb{E}^{\mathsf{P}}(1/N_{\sigma\wedge T}) = 1$. Again, we may assume w.l.o.g. that T is actually an $(\mathcal{F}_t)$-stopping time, cf. Lemma 4.3.1. Since $\sigma < T_0^Z = T_0^N \wedge T_0^D$ almost surely,

$$\begin{aligned}
\mathbb{E}^{\mathsf{P}}\left(\frac{1}{N_{\sigma\wedge T}}\right) &= \mathbb{E}^{\mathsf{P}}\left(\frac{\mathbb{1}_{\{T_0^Z>\sigma\}}}{N_{\sigma\wedge T}}\right) = \mathbb{E}^{\mathsf{P}}\left(\int_0^{T_0^Z}\frac{dA_u}{N_{u\wedge T}}\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(\int_0^{T_0^Z\wedge T}\frac{dA_u}{N_u} + \mathbb{1}_{\{T_0^Z>T\}}\frac{A_{T_0^Z}-A_T}{N_T}\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(\int_0^{T_0^Z\wedge T} D_u\frac{dA_u}{Z_u} + \mathbb{1}_{\{T_0^Z>T\}}\frac{Z_T}{N_T}\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(-\int_0^{T_0^Z\wedge T} D_u\frac{dD_u}{D_u} + \mathbb{1}_{\{T_0^Z>T\}}D_T\right)\\
&= \mathbb{E}^{\mathsf{P}}\left(1 - D_{T_0^Z\wedge T} + \mathbb{1}_{\{T_0^Z>T\}}D_T\right) = 1 - \mathbb{E}^{\mathsf{P}}\left(D_{T_0^Z}\mathbb{1}_{\{T_0^Z\leq T\}}\right)\\
&= 1 - \mathbb{E}^{\mathsf{P}}\left(D_{T_0^Z}\mathbb{1}_{\{T_0^N\leq T\}}\right) = 1 - \mathbb{E}^{\mathsf{P}}\left(D_\infty\mathbb{1}_{\{T_0^N\leq T\}}\right),
\end{aligned}$$

where in the last equality we used that $supp(dD) \subset \{Z > 0\}$, cf. Remark 4.4.5. Finally note that

$$\mathbb{E}^{\mathsf{P}}\left(D_\infty\mathbb{1}_{\{T_0^N\leq T\}}\right) = 0 \quad \Leftrightarrow \quad \mathsf{P}\left(T \geq T_0^N < T_0^D\right) = 0.$$

□

By taking $T = \infty$ in Theorem 4.6.9 we get

Corollary 4.6.10. *If $\mathsf{P}\left(T_0^N < T_0^D\right) = 0$, then NFLVR holds on the interval $[0,\sigma]$ with respect to the filtration $(\mathcal{G}_t)$.*

Remark 4.6.11. For an honest time σ the multiplicative decomposition of Z is given by $Z_t = N_t/\overline{N}_t$, where N is a non-negative local martingale converging to zero almost surely. And since a non-negative local martingale does not explode almost surely,

$$D_\infty = \frac{1}{\overline{N}_\infty} > 0 \quad \text{a.s.}$$

Therefore, $\mathsf{P}(T_0^D = \infty) = 1$ and $1/N_{t\wedge\sigma\wedge T}$ is a uniformly integrable martingale if and only if $\mathsf{P}(T_0^N \leq T) = \mathsf{P}(T_0^Z \leq T) = 0$. Therefore, if $T_0^N = \infty$ almost surely, $(1/N_{\sigma\wedge t})_{t\geq 0}$ is actually a true martingale and not a strict local martingale. In this case the fact that it is *not* uniformly integrable is already evident from the fact that $N_\sigma = \overline{N}_\infty > 1$ almost surely. Moreover, an application of Doob's maximal identity (cf. Lemma 2.1 in [59]) gives

$$\mathbb{E}^{\mathsf{P}}\left(\frac{1}{N_\sigma}\right) = \mathbb{E}^{\mathsf{P}}\left(\frac{1}{\overline{N}_\infty}\right) = \int_0^\infty \mathsf{P}\left(\frac{1}{\overline{N}_\infty} > x\right)dx = \int_0^1(1-x)dx = \frac{1}{2}.$$

Corollary 4.6.12. *Let T be a $(\mathcal{G}_t)$-stopping time. If either $\mathsf{P}(T_0^Z \leq T) = 0$ or $D_\infty = 0$ almost surely, then $NFLVR$ holds in the enlarged market on the time interval $[0, T \wedge \sigma]$.*

Proof. Note that the first claim is actually Theorem 4.6.5, but we can also derive it directly from Theorem 4.6.9: If $\mathsf{P}(T_0^Z \leq T) = 0$, then $\mathsf{P}(T \geq T_0^N) = \mathsf{P}(T \geq T_0^N \geq T_0^Z) = 0$. Moreover, if $D_\infty = D_{T_0^Z} = 0$ almost surely, then $\mathsf{P}(T_0^D \leq T_0^N) = 1$.
Hence, the claim follows from a combination of Theorem 4.6.9, Lemma 4.6.7 and Theorem 4.6.4. □

Of course, every pseudo-stopping time fulfills $D_\infty = 1 - A_\infty = 1 - 1 = 0$. The following example known as Émery's example shows that there are also other random times which allow for an equivalent local martingale measure up to time σ, even in a complete market.

Example 4.6.13. Let W be a $(\mathsf{P}, \mathcal{F}_t)$-Brownian motion and set $\sigma = \sup\{t \leq 1 : \ 2W_t = W_1\}$. The corresponding Azéma supermartingale is

$$Z_t = \sqrt{\frac{2}{\pi}} \int_{\frac{|W_t|}{\sqrt{1-t}}}^{\infty} x^2 e^{-x^2/2} dx = m_t - \sqrt{\frac{2}{\pi}} \int_0^t \frac{|W_u|}{(1-u)^{3/2}} \exp\left(-\frac{W_u^2}{2(1-u)}\right) du,$$

cf. [40]. For every $n \in \mathbb{N}$ define the set

$$B_n = \left\{ |W_u| > \sqrt{\frac{2}{n}} \ \forall \ u \in \left[1 - \frac{1}{n}, 1\right] \right\}$$

and note that

$$1 = \mathsf{P}(W_1 \neq 0) = \lim_{n \to \infty} \mathsf{P}(B_n).$$

On the set B_n we have for all $u \in \left[1 - \frac{1}{n}, 1\right]$,

$$\frac{|W_u|}{\sqrt{1-u}} > \sqrt{2}$$

and hence

$$\frac{1}{2} \int_{\frac{|W_u|}{\sqrt{1-u}}}^{\infty} x^2 e^{-x^2/2} dx \leq \int_{\frac{|W_u|}{\sqrt{1-u}}}^{\infty} (x^2 - 1) e^{-x^2/2} dx = \frac{|W_u|}{\sqrt{1-u}} \exp\left(-\frac{W_u^2}{2(1-u)}\right).$$

Thus, the following estimate holds on B_n:

$$\begin{aligned} \int_0^1 \frac{dA_t}{Z_t} \ &\geq \ \int_{1-\frac{1}{n}}^1 \frac{dA_t}{Z_t} = \int_{1-\frac{1}{n}}^1 \frac{dA_t}{\sqrt{\frac{2}{\pi}} \int_{\frac{|W_t|}{\sqrt{1-t}}}^{\infty} x^2 e^{-x^2/2} dx} \\ &\geq \ \frac{1}{2} \int_{1-\frac{1}{n}}^1 \frac{dA_t}{\sqrt{\frac{2}{\pi}} \frac{|W_t|}{\sqrt{1-t}} \exp\left(-\frac{W_t^2}{2(1-t)}\right)} = \frac{1}{2} \int_{1-\frac{1}{n}}^1 \frac{dt}{1-t} = \infty. \end{aligned}$$

Therefore, on each B_n we have

$$D_\infty = D_1 = \exp\left(-\int_0^1 \frac{dA_t}{Z_t}\right) = 0,$$

and by monotone convergence

$$\mathbb{E}^{\mathsf{P}}(D_\infty) = \lim_{n \to \infty} \mathbb{E}^{\mathsf{P}}(D_\infty \mathbb{1}_{B_n}) = 0 \quad \Leftrightarrow \quad D_\infty \equiv 0.$$

Unfortunately, the above results do only provide sufficient but not necessary conditions for NFLVR until time σ. In fact there may exist other local martingale deflators than $(1/N_{\sigma \wedge t})$ which could be uniformly integrable martingales. Even though the structure of all local martingale deflators can be derived as in [31], we cannot prove that they fail to be uniformly integrable martingales in general unless $N_\sigma > 1$ almost surely, which is e.g. the case for honest times, cf. Lemma 3.3 in [31].

4.7 Locally absolutely continuous change of measure

In this section we slightly change the general setup introduced in section 4.2.1. We will no longer rely on the existence of a random variable $\rho \geq 0$ to define Q, but instead we will only assume the existence of some non-negative $(\mathsf{P}, \mathcal{G}_t)$-martingale $(\widetilde{\rho}_t)$ with expectation one. As before (ρ_t) is the $(\mathcal{F}_t)$-optional projection of $(\widetilde{\rho}_t)$. Moreover, we will assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ is the natural augmentation of a probability space satisfying condition (P) introduced in [57].

For every $t \geq 0$ we now define a probability measure Q_t on $\mathcal{G}_t$ via $\mathsf{Q}_t = \widetilde{\rho}_t.\mathsf{P}|_{\mathcal{G}_t}$. This family of probability measures is consistent and since we assume our probability space to satisfy the natural (but not the usual!) assumptions, Corollary 4.9 of [57] yields the existence of a measure Q on $\mathcal{G}_\infty := \bigvee_{t\geq 0} \mathcal{G}_t$ such that $\mathsf{Q}|_{\mathcal{G}_t} = \mathsf{Q}_t$ for all $t \geq 0$. For convenience, let us assume that $\mathcal{F} = \mathcal{G}_\infty$ for the rest of this section. Note that Q is only locally absolutely continuous to P which we denote by $\mathsf{Q} \lhd \mathsf{P}$. We define the process h in this case by $h_t = \mathbb{E}^{\mathsf{P}}(\widetilde{\rho}_t \mathbb{1}_{\{\sigma>t\}}|\mathcal{F}_t)$. If $\mathsf{Q} \ll \mathsf{P}$, this definition coincides with the one in section 4.2.1. μ can now be defined as before.

In this setting the following slightly extended version of Theorem 4.2.1 holds.

Theorem 4.7.1. *Assume that $\mathsf{Q} \lhd \mathsf{P}$. If $U = (U_t)_{t\geq 0}$ is a local $(\mathsf{P}, \mathcal{F}_t)$-martingale, then the processes $X_t := \mathbb{1}_{\{\sigma>t\}} V_t \exp(\mu_t^F)$ and $V_{t\wedge\sigma}$ are both local $(\mathsf{Q}, \mathcal{G}_t)$-martingales, where $V_{t\wedge\sigma} := U_{t\wedge\sigma} - \langle U, \mu\rangle_{t\wedge\sigma}$.*

Proof. Since $\mathsf{Q}|_{\mathcal{G}_n} \ll \mathsf{P}|_{\mathcal{G}_n}$ the claim holds for every $U_t^n := U_{t\wedge n}$ according to Theorem 4.2.5. Especially, all processes are well-defined on $\bigcup_{n\in\mathbb{N}}[0, n] = \mathbb{R}_+$. But every process which is locally in $\mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t)$ is actually a local martingale on the whole time interval. $\square$

The motivation to study locally absolutely continuous changes of measures comes from the fact that it may allow us to get rid off the random time σ by pushing it to infinity as the following example demonstrates.

Example 4.7.2. Consider

$$\widetilde{\rho}_t = \frac{\mathbb{1}_{\{\sigma>t\}}}{Z_t^{\mathsf{P}}}.$$

This does indeed define a $(\mathcal{G}_t)$-martingale: For $s \leq t$,

$$\mathbb{E}^{\mathsf{P}}\left(\frac{\mathbb{1}_{\{\sigma>t\}}}{Z_t^{\mathsf{P}}}\middle|\,\mathcal{G}_s\right) = \frac{\mathbb{1}_{\{\sigma>s\}}}{Z_s^{\mathsf{P}}} \cdot \mathbb{E}^{\mathsf{P}}\left(\frac{\mathbb{1}_{\{\sigma>t\}}}{Z_t^{\mathsf{P}}}\middle|\,\mathcal{F}_s\right) = \frac{\mathbb{1}_{\{\sigma>s\}}}{Z_s^{\mathsf{P}}}.$$

Under the measure Q defined as above σ is pushed to infinity since

$$\mathsf{Q}(\sigma \leq t) = \mathbb{E}^{\mathsf{P}}\left(\widetilde{\rho}_t \mathbb{1}_{\{\sigma\leq t\}}\right) = \mathbb{E}^{\mathsf{P}}\left(\frac{\mathbb{1}_{\{\sigma>t\}}}{Z_t^{\mathsf{P}}}\mathbb{1}_{\{\sigma\leq t\}}\right) = 0 \quad \forall\, t \geq 0.$$

This is possible because $\widetilde{\rho}_t \to 0$ P-a.s. and therefore Q is not absolutely continuous to P on $\mathcal{F} = \mathcal{G}_\infty$. Thus, Q puts only positive weight on those events taking place before σ.
The fact that $\rho_t \equiv 1$ for all $t \geq 0$ implies that all $\mathcal{F}_t$-events do not "feel" the change of measure. Especially, any $(\mathsf{P}, \mathcal{F}_t)$-martingale is also a $(\mathsf{Q}, \mathcal{F}_t)$-martingale and by Theorem 4.7.1 also a $(\mathsf{Q}, \mathcal{G}_t)$-martingale because $h_t = \rho_t \equiv 1$.
Note that in computations of pre-σ events this measure change has the same impact as simply projecting down on $(\mathcal{F}_t)$. Indeed, every $\mathcal{G}_t$-measurable random variable is equal to an $\mathcal{F}_t$-measurable random variable before time σ, and for every $F_t \in \mathcal{F}_t$ one has

$$\mathbb{E}^{\mathsf{P}}(F_t \mathbb{1}_{\{\sigma>t\}}) = \mathbb{E}^{\mathsf{P}}\left(\frac{\mathbb{1}_{\{\sigma>t\}}}{Z_t^{\mathsf{P}}} \cdot F_t Z_t^{\mathsf{P}}\right) = \mathbb{E}^{\mathsf{Q}}(F_t Z_t^{\mathsf{P}}) = \mathbb{E}^{\mathsf{P}}(F_t Z_t^{\mathsf{P}}).$$

4.7.1 A change of measure which is equivalent to the enlargement formula

As before we denote by $Z^{\mathsf{P}} = N^{\mathsf{P}} D^{\mathsf{P}}$ the Itô-Watanabe decomposition of the Azéma supermartingale of σ. Under the assumption that N^{P} is a true martingale, we may set

$$\left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_{\mathcal{G}_t} = \widetilde{\rho}_t = \frac{\mathbb{1}_{\{\sigma>t\}}}{D_t^{\mathsf{P}}}.$$

One easily checks that this indeed defines a $(\mathcal{G}_t)$-martingale: For $s \leq t$,

$$\mathbb{E}^{\mathsf{P}}\left(\left.\frac{\mathbb{1}_{\{\sigma>t\}}}{D_t^{\mathsf{P}}}\right| \mathcal{G}_s\right) = \frac{\mathbb{1}_{\{\sigma>s\}}}{Z_s^{\mathsf{P}}} \cdot \mathbb{E}^{\mathsf{P}}\left(\left.\frac{\mathbb{1}_{\{\sigma>t\}}}{D_t^{\mathsf{P}}}\right| \mathcal{F}_s\right) = \frac{\mathbb{1}_{\{\sigma>s\}}}{Z_s^{\mathsf{P}}} \cdot \mathbb{E}^{\mathsf{P}}(N_t^{\mathsf{P}}|\mathcal{F}_s) = \frac{\mathbb{1}_{\{\sigma>s\}}}{D_s^{\mathsf{P}}}.$$

As in Example 4.7.2 we have $\mathsf{Q}(\sigma < \infty) = 0$ and hence any local $(\mathsf{Q}, \mathcal{F}_t)$-martingale is also a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale. However,

$$h_t = \rho_t = N_t^{\mathsf{P}}$$

is non-trivial and therefore the measure change will affect $(\mathsf{P}, \mathcal{F}_t)$-martingales according to the usual Girsanov theorem. Indeed, changing the measure in this way has the same effect as an application of the enlargement formula under P. This can be compared to [71], where the enlargement formula was derived by passing to the so called Föllmer measure associated with Z^{P}.

In this setup we have for any $\mathcal{F}_t$-measurable random variable F_t,

$$\mathbb{E}^{\mathsf{P}}(F_t \mathbb{1}_{\{\sigma>t\}}) = \mathbb{E}^{\mathsf{Q}}(F_t D_t^{\mathsf{P}}).$$

Since D^{P} is decreasing, one can interpret D_t^{P} as a discount factor in the above formula.

Remark 4.7.3. Note that in [14] the above measure change is applied to the valuation of defaultable securities via the reduced-form approach. However, in that paper the default time is directly modeled as a totally inaccessible stopping time without performing a progressive enlargement of filtration.

The following example provides some intuition why the above measure change pushes σ to infinity.

Example 4.7.4. Consider the honest time

$$\sigma = \sup\left\{t \geq 0 : \ N_t = \sup_{s\leq t} N_s\right\} = \sup\left\{t \geq 0 : \ \frac{1}{N_t} = \inf_{s\leq t} \frac{1}{N_s}\right\},$$

where N is supposed to be a non-negative $(\mathsf{P}, \mathcal{F}_t)$-martingale with $N_0 = 1$, converging towards zero almost surely. If we take $\widetilde{\rho}_t$ as above, the reciprocal of N becomes a Q-martingale: For $s \leq t$,

$$\mathbb{E}^{\mathsf{Q}}\left(\left.\frac{1}{N_t}\right| \mathcal{F}_s\right) = \frac{1}{\rho_s}\mathbb{E}^{\mathsf{P}}\left(\left.\frac{\rho_t}{N_t}\right| \mathcal{F}_s\right) = \frac{1}{N_s}.$$

However, $1/N$ does not converge to infinity but to zero under Q because Q is singular to P on $\mathcal{F}_\infty$. For all $\varepsilon > 0$ we have by dominated convergence as $t \to \infty$,

$$\mathsf{Q}\left(\frac{1}{N_t} > \varepsilon\right) = \mathbb{E}^{\mathsf{P}}\left(N_t \mathbb{1}_{\{1/\varepsilon > N_t\}}\right) \to 0.$$

Therefore, σ equals infinity almost surely under Q.

Remark 4.7.5. In the above computations we have assumed that N^{P} is a true martingale. If N^{P} is only a local $(\mathsf{P},\mathcal{F}_t)$-martingale, analogous computations can be done if one defines Q as the Föllmer measure associated with $(\widetilde{\rho}_t)$. In this case the random time σ is "replaced" under Q by the explosion time of $(\widetilde{\rho}_t)$, which equals the $(\mathcal{F}_t)$-stopping time $T_0^{D^{\mathsf{P}}}$ Q-almost surely.

4.8 An extension for honest times

So far we were only concerned with the time horizon $[0,\sigma]$. Of course, we cannot expect an analogue of Theorem 4.2.1 to hold after time σ because in general $(\mathcal{F}_t)$-semimartingales are not necessarily $(\mathcal{G}_t)$-semimartingales after time σ. Therefore, in this section we will assume that σ is an honest time. In this case it is well-known that the semimartingale property is preserved when passing from $(\mathcal{F}_t)$ to $(\mathcal{G}_t)$. Our goal is to proceed similarly to [56] in that we do not apply any results from the theory of enlargements of filtrations.

4.8.1 Change of measure after time σ

As before we assume that there exists a non-negative random variable ρ with expectation one and we set $\mathsf{Q}=\rho.\mathsf{P}$. We define the $(\mathsf{P},\mathcal{F}_t)$-submartingale k via

$$k_t=\mathbb{E}^{\mathsf{P}}(\rho|\mathcal{F}_t)-h_t=\mathbb{E}^{\mathsf{P}}\left(\rho\mathbb{1}_{\{\sigma\le t\}}|\mathcal{F}_t\right).$$

In the following we will use for fixed $u\ge 0$ the notation

$$\mathcal{M}^u(\mathsf{P},\mathcal{F}_t)$$

to denote the class of processes which are $(\mathsf{P},\mathcal{F}_t)$-martingales on the interval $[u,\infty)$. Moreover, for each $t\ge 0$ we choose an $\mathcal{F}_t$-measurable random variable σ_t which satisfies the requirement of Definition 4.3.7, i.e. $\mathbb{1}_{\{\sigma<t\}}\sigma=\mathbb{1}_{\{\sigma<t\}}\sigma_t$.

Lemma 4.8.1. *Fix $u\ge 0$ and let Y be an $(\mathcal{F}_t)$-adapted process such that $(\mathbb{1}_{\{\sigma_t\le u\}}k_tY_t)_{t\ge u}\in\mathcal{M}^u_{loc}(\mathsf{P},\mathcal{F}_t)$. Then $Y_t\mathbb{1}_{\{\sigma\le u\}}\in\mathcal{M}^u_{loc}(\mathsf{Q},\mathcal{G}_t)$.*

Proof. Because any $(\mathcal{F}_t)$-localizing sequence will also serve as a $(\mathcal{G}_t)$-localizing sequence, we only need to prove the martingale case. Recalling that σ is an honest time which avoids stopping times, we have for any bounded test function $F_s\in\mathcal{F}_s,\ s\le t$, and $u\le s\le t$,

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}(Y_t\mathbb{1}_{\{\sigma\le u\}}F_s) &= \mathbb{E}^{\mathsf{Q}}(Y_t\mathbb{1}_{\{\sigma\le t\}}\mathbb{1}_{\{\sigma_t\le u\}}F_s)=\mathbb{E}^{\mathsf{P}}(Y_t\rho\mathbb{1}_{\{\sigma\le t\}}\mathbb{1}_{\{\sigma_t\le u\}}F_s)\\
&= \mathbb{E}^{\mathsf{P}}(Y_tk_t\mathbb{1}_{\{\sigma_t\le u\}}F_s)=\mathbb{E}^{\mathsf{P}}(Y_sk_s\mathbb{1}_{\{\sigma_s\le u\}}F_s)\\
&= \mathbb{E}^{\mathsf{P}}(Y_s\rho\mathbb{1}_{\{\sigma\le s\}}\mathbb{1}_{\{\sigma_s\le u\}}F_s)=\mathbb{E}^{\mathsf{Q}}(Y_s\mathbb{1}_{\{\sigma\le u\}}F_s).
\end{aligned}$$

Furthermore, if in addition $r\le s$, then one gets

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}(Y_t\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma\le r\}}F_s) &= \mathbb{E}^{\mathsf{Q}}(Y_t\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma\le s\}}\mathbb{1}_{\{\sigma_s\le r\}}F_s)=\mathbb{E}^{\mathsf{Q}}(Y_t\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma_s\le r\}}F_s)\\
&= \mathbb{E}^{\mathsf{Q}}(Y_s\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma_s\le r\}}F_s)=\mathbb{E}^{\mathsf{Q}}(Y_s\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma\le s\}}\mathbb{1}_{\{\sigma_s\le r\}}F_s)\\
&= \mathbb{E}^{\mathsf{Q}}(Y_s\mathbb{1}_{\{\sigma\le u\}}\mathbb{1}_{\{\sigma\le r\}}F_s).
\end{aligned}$$

The monotone class theorem allows us to conclude that $Y_t\mathbb{1}_{\{\sigma\le u\}}$ is a Q-martingale with respect to $\left(\mathcal{F}_t\vee\sigma(\mathbb{1}_{\{\sigma\le r\}};r\le t)\right)_{t\ge u}$. Because martingales with respect to some filtration remain martingales with respect to its right-continuous augmentation, we thus conclude that $Y_t\mathbb{1}_{\{\sigma\le u\}}\in\mathcal{M}^u(\mathsf{Q},\mathcal{G}_t)$. □

Remark 4.8.2. Note that if $(Y_t)_{t\geq u}$ is a martingale with respect to Q and $(\mathcal{G}_t)_{t\geq u}$ on the set $\{\sigma \leq u\}$, then it is also a martingale on any $\mathcal{G}_u$-measurable subset of $\{\sigma \leq u\}$. Thus, for example

$$(Y_t \mathbb{1}_{\{u_i \leq \sigma < u_j\}})_{t\geq u} \in \mathcal{M}^u(\mathcal{G}_t, \mathsf{Q})$$

for every $0 \leq u_i < u_j \leq u$.

Lemma 4.8.3. *Let $Y = (Y_t)_{t\geq 0}$ be a process such that $(\mathbb{1}_{\{\sigma\leq u\}}(Y_{t\vee u} - Y_u))_{t\geq 0} \in \mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t)$ for all $u > 0$. Then $Y_{t\vee\sigma} - Y_\sigma \in \mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t)$.*

Proof. Let us first assume that Y is bounded. Then by Remark 4.8.2 for all $u > v \geq 0$,

$$\left(\mathbb{1}_{\{v\leq\sigma<u\}}(Y_{t\vee u} - Y_u)\right)_{t\geq 0} \in \mathcal{M}(\mathsf{Q}, \mathcal{G}_t).$$

We approximate σ with the decreasing sequence of $(\mathcal{G}_t)$-stopping times

$$s_n := \sum_{k=1}^{n2^n} \frac{k}{2^n} \mathbb{1}_{\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}} + \infty \mathbb{1}_{\{\sigma\geq n\}},$$

taking only finitely many values. Then for $s \leq t$ and $G_s \in \mathcal{G}_s$, because Y is assumed to be bounded and càdlàg,

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}\left((Y_{t\vee\sigma} - Y_\sigma)\mathbb{1}_{G_s}\right) &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left((Y_{t\vee s_n} - Y_{s_n})\mathbb{1}_{G_s}\right) \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\sum_{k=1}^{n2^n} \mathbb{1}_{\{s_n=k2^{-n}\}}(Y_{t\vee(k2^{-n})} - Y_{k2^{-n}})\mathbb{1}_{G_s}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n} \mathbb{E}^{\mathsf{Q}}\left(\underbrace{\mathbb{1}_{\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}}(Y_{t\vee(k2^{-n})} - Y_{k2^{-n}})}_{\in\mathcal{M}(\mathsf{Q},\mathcal{G}_t)}\mathbb{1}_{G_s}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n2^n} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{(k-1)2^{-n}\leq\sigma<k2^{-n}\}}(Y_{s\vee(k2^{-n})} - Y_{k2^{-n}})\mathbb{1}_{G_s}\right) \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\sum_{k=1}^{n2^n} \mathbb{1}_{\{s_n=k2^{-n}\}}(Y_{s\vee(k2^{-n})} - Y_{k2^{-n}})\mathbb{1}_{G_s}\right) \\
&= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left((Y_{s\vee s_n} - Y_{s_n})\mathbb{1}_{G_s}\right) = \mathbb{E}^{\mathsf{Q}}\left((Y_{s\vee\sigma} - Y_\sigma)\mathbb{1}_{G_s}\right),
\end{aligned}$$

which proves that $Y_{t\vee\sigma} - Y_\sigma \in \mathcal{M}(\mathsf{Q}, \mathcal{G}_t)$. Now the general case follows by localization of Y. □

Theorem 4.8.4. *Let σ be an honest time and suppose that $(U_t)_{t\geq 0}$ is local $(\mathsf{P}, \mathcal{F}_t)$-martingale. Then the process*

$$V_t := U_t + \int_0^{\sigma\wedge t} \frac{d\langle U, h\rangle_s}{h_s} - \int_\sigma^{\sigma\vee t} \frac{d\langle U, k\rangle_s}{k_s}$$

is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale.

Proof. From Theorem 4.2.5 we already know that $(V_{t\wedge\sigma})_{t\geq 0}$ is a local $(\mathsf{Q}, \mathcal{G}_t)$-martingale. Therefore, it remains to show that $V_{t\vee\sigma} - V_\sigma \in \mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t)$. According to Lemma 4.8.3 this holds if for all $u > 0$,

$$\mathbb{1}_{\{\sigma\leq u\}}(V_{t\vee u} - V_u) = \mathbb{1}_{\{\sigma\leq u\}}V_t^u \in \mathcal{M}_{loc}(\mathsf{Q}, \mathcal{G}_t) \qquad \Leftrightarrow \qquad \mathbb{1}_{\{\sigma\leq u\}}V_t^u \in \mathcal{M}_{loc}^u(\mathsf{Q}, \mathcal{G}_t),$$

where we have defined for each $s \in \mathbb{R}_+$ the $(\mathcal{F}_t)$-adapted process

$$V_t^s := U_{t\vee s} - U_s - \int_s^{t\vee s} \frac{d\langle k, U\rangle_u}{k_u}.$$

Therefore, an application of Lemma 4.8.1 will yield the result, if we can show that for all $u \geq 0$,

$$(\mathbb{1}_{\{\sigma_t \leq u\}} k_t V_t^u)_{t\geq u} \in \mathcal{M}_{loc}^u(\mathsf{P}, \mathcal{F}_t).$$

First note that

$$m_t^u := \mathbb{1}_{\{\sigma_t\leq u\}} k_t = \mathbb{E}^{\mathsf{P}}(\rho \mathbb{1}_{\{\sigma\leq t, \sigma_t \leq u\}}|\mathcal{F}_t) = \mathbb{E}^{\mathsf{P}}(\rho \mathbb{1}_{\{\sigma \leq u\wedge t\}}|\mathcal{F}_t)$$

and hence for every fixed $u > 0$, $m^u \in \mathcal{M}^u(\mathsf{P}, \mathcal{F}_t)$. We apply integration by parts for $t \geq u$ to get

$$\begin{aligned} d(\mathbb{1}_{\{\sigma_t\leq u\}} k_t V_t^u) &= d(m_t^u V_t^u) = V_t^u dm_t^u + m_t^u dV_t^u + d\langle m^u, V^u\rangle_t \\ &= V_t^u dm_t^u + \mathbb{1}_{\{\sigma_t \leq u\}} \left[k_t \left(dU_t - \frac{d\langle k, U\rangle_t}{k_t} \right) + d\langle k, U\rangle_t \right] \\ &= V_t^u dm_t^u + m_t^u dU_t, \end{aligned}$$

which is an element of $\mathcal{M}_{loc}^u(\mathsf{P}, \mathcal{F}_t)$ for every $u > 0$ as required. □

Remark 4.8.5. In fact a more general version of Theorem 4.8.4 is known to hold even without assuming (AC). This can be proven by applying first Girsanov's theorem and second the enlargement formula for honest times as it is done in paragraph 81 in [22]. Note however, that our proof does *not* make use of the enlargement formula. It only uses Definition 4.3.7 of an honest time. Therefore as a byproduct by setting $\rho \equiv 1$ we do actually recover the enlargement formula after σ for honest times.

Example 4.8.6. *(Continuation of Example 4.3.12)* We set $\sigma_t = \sup\left\{u \leq t \wedge T_1^B : \ B_u = 0\right\}$.

$$\begin{aligned} k_t &= \mathbb{E}^{\mathsf{P}}\left(f(\overline{B}_\pi)\mathbb{1}_{\{\sigma\leq t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(f(\overline{B}_\sigma)\mathbb{1}_{\{\sigma\leq t\}}|\mathcal{F}_t\right) = \mathbb{E}^{\mathsf{P}}\left(f(\overline{B}_{\sigma_t})\mathbb{1}_{\{\sigma\leq t\}}|\mathcal{F}_t\right) \\ &= f(\overline{B}_{\sigma_t})(1 - Z_t^\sigma) = f(\overline{B}_{\sigma_t}) B^+_{t\wedge T_1^B} \\ dk_t &= f(\overline{B}_{\sigma_t}) \left(\mathbb{1}_{\{B_t>0\}} dB_{t\wedge T_1^B} + \frac{1}{2} dL_{t\wedge T_1^B} \right), \end{aligned}$$

where we used the fact that $supp(d\sigma_t) \subset \{B = 0\}$. Hence, according to Theorem 4.8.4 the process

$$W_t := B_t + \int_0^{t\wedge\sigma} \frac{\mathbb{1}_{\{B_t>0\}} f(\overline{B}_t) dt}{\int_{\overline{B}_t}^1 f(y)dy + f(\overline{B}_t)(\overline{B}_t - B_t^+)} + \int_{t\wedge\sigma}^{t\wedge T_1^B} \frac{dt}{B_t}$$

is a $(\mathsf{Q}, \mathcal{G}_t)$-Brownian motion. The result is not surprising, of course, since $\rho = \overline{B}_\pi = \overline{B}_\sigma \in \mathcal{G}_\sigma$. Therefore the measure change has no effect after σ and we do indeed recover the usual term from the enlargement formula under P on the interval $[\sigma \wedge t, t]$. Note that the same effect will appear when dealing with the generalization of this example from Section 4.3.3.

In the next subsection we will provide a more interesting example.

4.8.2 Relative martingales

Relative martingales were introduced in [6]. We will work with

Definition 4.8.7. Let σ be an honest time and (Y_t) an $(\mathcal{F}_t)$-adapted right-continuous process such that $Y_\infty := \lim_{t\to\infty} Y_t$ exists P-almost surely and in $L^1(\mathsf{P})$. Then (Y_t) is called a relative martingale associated with σ, if $Y_t = \mathbb{E}^{\mathsf{P}}(Y_\infty \mathbb{1}_{\{\sigma\le t\}}|\mathcal{F}_t)$ for all $t \ge 0$.

Note that for an honest time σ the process $k_t = \mathbb{E}^{\mathsf{P}}(\rho\mathbb{1}_{\{\sigma\le t\}}|\mathcal{F}_t)$ introduced in the last subsection is a relative martingale with final value $k_\infty = \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{F}_\infty)$. Therefore, the class of relative martingales associated with σ will provide us with nice non-trivial examples to illustrate Theorem 4.8.4. The following result from [6] is very helpful in finding relative martingales.

Lemma 4.8.8. *Let (Y_t) be a continuous non-negative submartingale of class (D) with Doob-Meyer decomposition $Y = M + F$, where $M \in \mathcal{M}_{loc}(\mathsf{P}, \mathcal{F}_t)$ and F is an increasing $(\mathcal{F}_t)$-adapted process. Assume that $M_0 = F_0 = 0$, $\mathsf{P}(Y_\infty = 0) = 0$ and that the measure (dF_t) is carried by the set $\{t:\ Y_t = 0\}$. Then (Y_t) is a relative martingale associated with $\sigma = \sup\{t \ge 0:\ Y_t = 0\}$.*

Example 4.8.9. Let B be a standard $(\mathsf{P}, \mathcal{F}_t)$-Brownian motion with L denoting its local time at level zero. Set $\sigma = \sup\{\sigma \le 1:\ B_t = 0\}$. The submartingale

$$|B_{t\wedge 1}| = \int_0^{t\wedge 1} \operatorname{sgn}(B_u)dB_u + L_{t\wedge 1}$$

fulfills the assumptions of Lemma 4.8.8 and is hence a relative martingale associated with σ. Setting $\rho = |B_1|$ we have for $t \le 1$,

$$\begin{aligned}
k_t &= |B_t| = \int_0^t \operatorname{sgn}(B_u)dB_u + L_t \\
\rho_t &= \mathbb{E}^{\mathsf{P}}(\rho|\mathcal{F}_t) = \mathbb{E}^{\mathsf{P}}(|B_1||\mathcal{F}_t) = \int_{-\infty}^{\infty} \frac{|x+B_t|}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{x^2}{2(1-t)}\right) dx \\
&= |B_t| \cdot \left[2\Phi\left(\frac{|B_t|}{\sqrt{1-t}}\right) - 1\right] + \sqrt{\frac{2(1-t)}{\pi}} \cdot \exp\left(-\frac{|B_t|^2}{2(1-t)}\right) \\
h_t &= \rho_t - k_t = 2|B_t| \cdot \left[\Phi\left(\frac{|B_t|}{\sqrt{1-t}}\right) - 1\right] + \sqrt{\frac{2(1-t)}{\pi}} \cdot \exp\left(-\frac{|B_t|^2}{2(1-t)}\right) \\
dh_t &= 2\left[\Phi\left(\frac{|B_t|}{\sqrt{1-t}}\right) - 1\right] \operatorname{sgn}(B_t)dB_t + \text{finite variation part.}
\end{aligned}$$

Thus according to Theorem 4.8.4 the process

$$W_t := B_t - \int_0^{t\wedge\sigma} \frac{\operatorname{sgn}(B_s)\left[\Phi\left(\frac{|B_s|}{\sqrt{1-s}}\right) - 1\right] ds}{|B_s| \cdot \left[\Phi\left(\frac{|B_s|}{\sqrt{1-s}}\right) - 1\right] + \sqrt{\frac{1-s}{2\pi}} \cdot \exp\left(-\frac{|B_s|^2}{2(1-s)}\right)} + \int_{t\wedge\sigma}^{t\wedge 1} \frac{ds}{B_s}$$

is a $(\mathsf{Q}, \mathcal{G}_t)$-Brownian motion.

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